Matching with Overlapping Quotas under Weak Preferences

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Abstract

We develop a framework for analyzing a school choice environment where each student may simultaneously satisfy multiple quota criteria, such as race and gender. In our environment, schools primarily care about filling as many quota seats as possible and then care about maximizing student quality. Students have weak preferences over their admitted status at a school, allowing students to either prefer one admitted status to another or be indifferent between two admitted statuses. By considering the sets of contracts that students are indifferent between as the main object in the school's choice function, we present a method for analyzing the overlapping quota problem. Our procedure can be nested into a cumulative offer process and has appealing computational properties.

Keywords: quotas, school choice, cumulative offer, matching, affirmative action
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1. Introduction

In school choice, the standard approach is to give equal consideration to all students, regardless of their background. However, schools and policymakers may want to accomplish specific social objectives, such as increasing the number of admitted students in underrepresented groups. Policymakers may use group-specific quotas to increase the number of admitted students from underrepresented groups, using multiple sets of quotas to increase the representation of all of the groups simultaneously. If each student has at most one privileged status, the theory of how to incorporate quotas into matching is well-established. Quota seats and non-quota seats are separated, and students with a privileged status are given priority for the quota seats (Abdulkadiroğlu, 2005). Any seats that cannot be filled by privileged students are made available to general students (Hafalir et al., 2013). The student-optimal stable allocation can be determined using a cumulative offer process[1] where privileged students have higher priority over seats assigned to their privileged status.

When students can hold multiple privileged statuses, the problem is more complicated. Students could be assigned to any one of these statuses at the same school, and may either be indifferent between these statuses or may prefer being admitted under a specific status. The overlapping quota problem has been solved for two polar cases, when students are indifferent to their admitted status (Hatfield and Milgrom, 2005; Westkamp, 2013) and when students have a strict preference over admitted status (Aygün and Turhan, 2017a). However, when students have weak preferences over admitted statuses, neither method is applicable. In this manuscript, we provide a framework for dealing with weak preferences that considers the sets of contracts that students are indifferent between. Given the set of contracts, we determine the school's choice correspondence by iteratively determining whether it is beneficial to add new students. Using this approach, we find the student-optimal stable allocation using a cumulative offer process where students propose sets of contracts that they are indifferent between and schools accept or reject each student's sets of contracts.

It is natural to assume students have weak preferences over their admitted status. When there are no additional advantages of being admitted under a particular status, students are likely indifferent between being admitted under a different status at the same school. In college admissions in America, for instance, many schools have affirmative action programs that do not provide additional funding based on a student's admitted status. A student

[1] The cumulative offer process is a generalization of the deferred acceptance algorithm (Hatfield and Milgrom, 2005). We use this to refer to both the specific incidences of the deferred acceptance algorithm and the more general cumulative offer process.

that is acceptable under multiple statuses is unlikely to care about her admitted status. However, if funding opportunities depend on the admitted status, students may have strict preferences over their admitted status. In India, the Indian Institutes of Technology admission procedure uses quotas to mitigate historical discrimination faced by disadvantaged castes (Aygün and Turhan, 2017b). As the admitted status carry different opportunities for financial assistance, students may have strict preferences over their admitted statuses.[2]

When students are indifferent to their admitted status, Hatfield and Milgrom (2005) characterize the school's choice problem as a linear maximization problem. Each student has a value that depends on her quality and whether she holds the status of her assigned seat. The school chooses a set of students that maximizes the total value of its seats when student are optimally assigned to statuses. The associated choice function satisfies substitutability and irrelevance of rejected contracts; therefore, stable allocations can be found using a cumulative offer process. However, substitutability is satisfied because each contract includes only a student and a school. When students have weak preferences over statuses, contracts must contain an admitted status, so substitutability no longer holds.

Westkamp (2013) provides an alternative mechanism for students that are indifferent to their admitted status. Each status has a capacity and a preference ordering over students with the same status. Statuses are filled sequentially, and any unused capacity is transferred to other statuses. Although the capacity transfer mechanism provides a versatile way to transfer unused capacity from one status to another, students are considered for statuses sequentially. When the mechanism accepts a student, it does not consider her value under other statuses. Therefore, the mechanism may produce unstable allocations. Further, it may give favorable treatment for students of one status at the expense of other students.

Alternatively, when students have strict preferences over their admitted status, Aygün and Turhan (2017a) consider contracts that include a student, school, and admitted status.[3] They use a monotone capacity transfer mech-

[2]Students are first considered for general seats, so they may have an incentive to reduce their test score. Students with a privileged status have been known to target their score over a specific range in an attempt to influence their admitted status.

[3]Aygün and Turhan (2017a) are motivated by the admission process for engineering schools in India. The system grants priority to various minority groups that overlap. Fur-

anism, which provides a process for transferring unused capacity to other students. Although the school's choice function no longer satisfies substitutability, weaker conditions are satisfied which ensure stable allocations exist and allows a cumulative offer process to find them. Under their structure, the student-optimal cumulative offer process is stable and strategy-proof, and it respects improvement in priorities.

Although the method of Aygün and Turhan (2017a) performs well when students have strict preferences over their admitted status, issues arise when students have weak preferences.[4] When students are indifferent between being admitted under two different statuses, the students' submitted preferences could lead to allocations that are not stable. If there is a systematic tendency to order one status over another, students with multiple statuses may tend to rank being admitted under one status higher than others,[5] making getting accepted more difficult for students with that status but easier for students with other statuses.

We develop an alternative framework that accommodates students with weak preferences towards their admitted status. In our environment, students have weak preferences over allocations with the same school but have strict preferences over allocations with different schools. Schools' preferences over students consist of two parts. Their primary objective is to fill as many privileged seats as possible. Their secondary objective is to maximize student quality. Specifically, schools prefer allocations that fill more privileged seats to allocations that fill fewer privileged seats. When two allocations fill the same number of privileged seats, student quality is maximized considering the responsive preferences induced by the school's ranking of individual students (Roth, 1985).

Similar preference structures are common in school choice, especially in the form of quotas. India and Brazil use class and race-based quotas as part

ther, different scholarships may be available to students based on their admitted statuses.

[4]The literature on matching with indifference focuses on when schools are indifferent between the accepted student and finds that random tie breaking can lead to inefficiency (Erdil and Ergin, 2008; Abdulkadiroğlu et al., 2009). In our environment, students are indifferent whereas schools have strict preferences. We discuss these details further in the conclusion.

[5]When students are to state strict preferences over statuses, they may find reasons to list preference in one way or another. Students of a particular status that have an affinity for other group members may order the affinity status lower than other statuses, benefiting the affinity students while harming others.

of their university admissions process. The German university admissions process has quotas that give priority to top-ranked high school students and students returning after a period of absence (Westkamp, 2013). Our preferences apply readily to these environments. Additionally, the preferences could be adopted in place of a slot-specific priorities (Kominers and Sönmez, 2016). In Boston, students that are in walk zones and have siblings that go to the school are given priority for a portion of each school's available seats (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu et al., 2005). The highest priority is given to students that are both in the walk zone and have siblings that go to the school. We provide an alternative to slot-specific priorities that does not require a policymaker decide which status should have a higher priority, thereby making it easier to add additional privileged statuses.

We extend the matching with contracts framework (Hatfield and Milgrom, 2005) to account for indifference by using choice correspondences instead of choice functions. The choice correspondences give the set of maximal allocations for students and schools, so students and schools are indifferent between every allocation in their choice correspondence. We determine the school's choice correspondence for both hard and soft bounds. Under hard bounds, any empty privileged seats remain vacant. Under soft bounds, any empty privileged seats are available to general students. The theory behind hards bounds is more elegant and has a simpler algorithm for determining the school's choice correspondence; however, it is also necessary to consider soft bounds due to the inefficiency resulting from the empty seats that may occur under hard bounds. For soft bounds, we use minority reserves due to the adverse effect that majority quotas can have on the privileged groups (Kojima, 2012; Hafalir et al., 2013).

Using the choice correspondence, we characterize stability by allowing blocking coalitions to include students that are indifferent between the original allocation and the new allocation. In our environment, contracts that are not in the school's choice correspondence may be in the choice correspondence when a new student is added; therefore, we focus on feasible sets of contracts: sets of contracts where all the students in the contracts can be allocated to seats at the school. Given a feasible set of contracts, we determine the set of students that a student can replace when she is added to the allocation. By iteratively considering each student and determining whether she is better than the minimum ranked student she can replace, we determine the students in the school's choice correspondence.

Given this framework, we develop the adaptive assignment algorithm.

Students provide a ranking over sets of contracts, where each set of contracts includes a set of contracts that the student is indifferent between. Students start by applying to their top-ranked school with the first set of contracts. Given the set of contracts available to the school, the school chooses an optimal set of contracts and conditionally accepts the students associated with the contracts. The school retains all of the contracts of those students. Any student that does not have an acceptable contract is rejected and applies to the next school with the next set of contracts. This algorithm proceeds until all the students are either matched with a school or have no contracts remaining.

Our algorithm has similar properties to the deferred acceptance algorithm introduced in Gale and Shapley (1962). Students are considered on a one-by one-basis and conditionally accepted if they are preferred to minimum ranked student they can replace. Our process is linear in the number of students. Alternatively, the process in Hatfield and Milgrom (2005) for determining the student-optimal stable allocation grows exponentially as the number of seats at a school increases.

We proceed as follows: we develop the model in Section 2. In Sections 3 and 4, we develop an algorithm for determining the choice correspondence under hard bounds and soft bounds, respectively. In Section 5, we look at the properties of the choice correspondence and show that a stable allocation exists. In Section 6, we present the adaptive assignment algorithm. Finally, Section 7 concludes by relating our results to the relevant literature.[6]

2. Model

We use the matching with contracts framework from Hatfield and Milgrom (2005), adopting notation similar to Aygün and Turhan (2017a). We assume that there are a finite number of students $I = \{i_1, \ldots, i_n\}$ and schools $S = \{s_1, \ldots, s_m\}$. There are a finite set of statuses $T = \{t_0, t_1, \ldots, t_k\}$ that can be held by students. We let t_0 be a general status that is held by all students; therefore, the set of privileged statuses is given by $T^p = T/\{t_0\}$. The statuses held by the students is given by the correspondence $\tau : I \Rightarrow T$, where $\tau(i)$ is the set of statuses held by student i.

[6] A survey of the early matching literature is Roth and Sotomayor (1992). Sönmez and Ünver (2011) survey more recent developments.

There are a finite set of possible contracts $X \subseteq I \times S \times T$. Each contract $x \in X$ includes a student $i(x)$, school $s(x)$, and assigned status $t(x)$. Students must hold the status assigned by the contract; therefore, the possible contracts are $X = \{x \in X : t(x) \in \tau(i(x))\}$. Given a set of contracts $Y \subseteq X$, the students, schools, and assigned status of Y are given by $i(Y) \equiv \cup_{x\in Y}\{i(x)\}$, $s(Y) \equiv \cup_{x\in Y}\{s(x)\}$, and $t(Y) \equiv \cup_{x\in Y}\{t(x)\}$, respectively. For $Y \subseteq X$, the set of contracts including student i, school s, and student t are $Y_i \equiv \{x \in Y : i(x) = i\}$, $Y_s \equiv \{x \in Y : s(x) = s\}$, and $Y_t \equiv \{x \in Y : t(x) = t\}$, respectively. We define sets of contracts that include both student i and school s by $Y_{i,s} = Y_i \cap Y_s$. Similarly, $Y_{s,t} = Y_s \cap Y_t$. Finally, the contract containing a set of student $I' \subseteq I$ is $Y_{I'} = \{x \in Y : i(x) \in I'\}$.

The capacity of school s is given by $\bar{q}^s$. School s's quota for privileged status $t \in T^p$ is $\bar{q}^s_t$. Quota seats are reserved for students of their respective statuses, so $\sum_{t\in T^p} \bar{q}^s_t \leq \bar{q}^s$. Under hard bounds, any unfilled privilege seats are unavailable to other students, the remaining $\bar{q}_{t_0} = \bar{q}^s - \sum_{t\in T^p} \bar{q}^s_t$ seats are available for general students. For hard bounds, an allocation is an $X' \subseteq X$ such that $|X'_i| \leq 1$ for all $i \in I$ and $|X'_s \cap X'_t| \leq \bar{q}^s_t$ for all $s \in S$ and $t \in T$. Under soft bounds, seats that are not allocated to a privileged student are available for general students. For soft bounds, an allocation is an $X' \subseteq X$ such that $|X'_i| \leq 1$ for all $i \in I$, $|X'_s| \leq \bar{q}^s$ for all $s \in S$, and $|X'_s \cap X'_t| \leq \bar{q}^s_t$ for all $s \in S$ and $t \in T^p$. The set of allocations is denoted by $\mathcal{X}$. The set of allocations containing school s is denoted by $\mathcal{X}_s$.

2.1. Students' Preferences

Deviating from Aygün and Turhan (2017a), we assume that student i has a preference relation $\succsim_i$ over X_i that has weak preferences over contracts with the same school, but has strict preferences over contracts with different schools. Formally, for $x, x' \in Y_i$, if $s(x) \neq s(x)$ then $x \nsim_i x'$. A student that prefers to be unmatched to being allocated contract x has $\emptyset \succ_i x$. Additionally, we require that students weakly prefer each privileged status to the general status for each school: $x' \succsim_i x$ for any $x, x' \in X$, such that $i(x) = i(x')$, $s(x) = s(x')$, and $t(x) = t_0$. This assumption is unnecessary from a theoretical standpoint; however, students strictly preferring the general status can lead to over-representation of privileged students.[7]

The students' preferences allow them to be indifferent between alloca-

[7]We show this in Example 8 of Appendix B.

tions that include the same school but have different statuses. Therefore, we use choice correspondences for the students. The choice correspondence for student i is the set of contracts that include student i and are maximal for student i under $\succsim_i$:

$$\mathcal{C}^i(Y) = \{x \in Y_i : s(x) \succsim_i s' \text{ for all } s' \in s(Y_i) \cup \emptyset\}$$

Because students strict have preferences over contracts with different schools, $s(x) = s(x')$ for all $x, x' \in \mathcal{C}^i(Y)$.

2.2. Schools' Preferences

Schools have lexicographical preferences over allocations. Their primary objective is to choose an allocation that maximizes the number of acceptable students allocated to privileged seats. When comparing two allocations that fill the same number of privileged seats, schools prefer the allocation that maximizes student quality. Schools have a strict priority order over pairs of students, denoted by π^s, that is independent of the students' statuses. Student i has a higher priority than student i' at school s if $i \, \pi^s \, i'$. Student i is acceptable to school s if $i \, \pi^s \, \emptyset$ and is unacceptable if $\emptyset \, \pi^s \, i$. When choosing between two allocations that fill the same number of privileged seats the students preferences are responsive with respect to π^s.

Then, defining $q^s_p(X') = |\{x \in X'_{T^p} : i(x) \succ_s \emptyset\}|$, an allocation $X' \succsim_s X''$ if either:

- $q^s_p(X') > q^s_p(X'')$, or
- $q^s_p(X') = q^s_p(X'')$, $|X'| \geq |X''|$, and there are non-repeating sequences $x_1, \ldots, x_{|X''|} \in X'$ and $x'_1, \ldots, x'_{|X''|} \in X''$ such that $i(x_l) \, \pi^s \, i(x'_l)$ or $i(x_l) = i(x'_l)$ for all $l \in \{1, \ldots, |X''|\}$.

An allocation $X' \succsim_s X''$ if either X' fill more quota seats or if they fill the same number of quota seats and each students in X'' can be paired with a unique student in X', such that the students in X' are pairwise preferred to the students in X''. The school's preferences induce a preorder $\succsim_s$ on $\mathcal{X}_s$.[8] The example below shows that some allocations are comparable whereas others are not.

[8] A preorder satisfies reflexivity (i.e. $X' \succsim X'$) and transitivity (i.e. $X' \succsim X''$ and $X'' \succsim X'''$ imply $X' \succsim X'''$). We show that our construction is a preorder in Appendix C.

Example 1. Assume $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s\}$,[9] and $T = \{t_0, t_1\}$. Assume $\bar{q} = 2$ with $\bar{q}_{t_1} = 1$. Let $i_1 \,\pi\, i_2 \,\pi\, i_3 \,\pi\, i_4$. Let $Y = \{(i_1, t_0), (i_2, t_0), (i_3, t_1), (i_4, t_1)\}$. The contracts $X' = \{(i_1, t_0), (i_4, t_1)\}$ and $X'' = \{(i_2, t_0), (i_3, t_1)\}$ are not comparable. Neither $X' \succsim X''$ nor $X'' \succsim X'$. However, there exists a X''' such that $X''' \succsim X'$ and $X''' \succsim X''$, namely $X''' = \{(i_1, t_0), (i_3, t_1)\}$.

We use $\succsim_s$ to characterize a choice correspondence for s given a set of contract Y. An allocation $X' \subseteq Y_s$ is in the schools choice correspondence if it is maximal under the preorder $\succsim_s$. Although, allocations need not be comparable, we show in Propositions 4 and 11 that any allocation in the choice correspondence of Y is weakly preferred to any other allocations contained in Y. Therefore, the choice correspondence for school s:

$$\mathcal{C}^s(Y) = \{X' \in Y_s : X' \succsim_s X'' \;\forall\; X'' \in Y_s\}$$

is well-defined and includes all maximal elements of Y_s under the preorder $\succsim_s$. Further, $i(X') = i(X'')$ for all $X', X'' \in \mathcal{C}^s(Y)$.[10]

2.3. Equilibrium

We focus on stable allocations, allocations that are individually rational (IR) and unblocked (UB). An allocation is individually rational when the students and schools find their contracts acceptable. An allocation is unblocked when there is no allocation that is strictly preferred to the original allocation by a school and weakly preferred to the original allocation by the new students allocated to the school.

Definition 1. An allocation X' is **stable** if it is IR and UB:

IR: $X'_i \in \mathcal{C}^i(X')$ for all $i \in I$ and $X'_s \in \mathcal{C}^s(X')$ for all $s \in S$.

UB: There is no $s \in S$ and $X'' \subseteq X_s$ such that $X'' \in \mathcal{C}^s(X' \cup X'')$, $X'_s \notin \mathcal{C}^s(X' \cup X'')$, and $X''_i \in \mathcal{C}^i(X' \cup X'')$ for $i \in i(X'')$.

[9] As there is only one school, we omit the subscripts and superscripts referring to school s. We also omit s from each contract in sets of contracts. We do the same in other examples that include single school.

[10] These results are also shown in Propositions 4 and 11.

Stability accounts for weak preferences by allowing for blocking coalitions where students are indifferent between the original allocation and a new allocation. The following example illustrates that a student in a blocking coalition can be indifferent between the original allocation and a new allocation.

Example 2. Let $I = \{i_1, i_2, i_3\}$, $S = \{s\}$, and $T = \{t_0, t_1\}$. Let $\bar{q}^s = 2$ and $\bar{q}^s_{t_1} = 1$. Assume $\tau(i_1) = \tau(i_3) = \{t_0, t_1\}$ and $\tau(i_2) = \{t_0\}$. The school's preferences are $i_1 \, \pi^s \, i_2 \, \pi^s \, i_3$. The students' preferences are $(i_1, t_0) \sim_{i_1} (i_1, t_1) \succ_{i_1} \emptyset$, $(i_2, t_0) \succ_{i_2} \emptyset$, and $(i_3, t_1) \succ_{i_3} (i_3, t_0) \succ_{i_3} \emptyset$. Then $X' = \{(i_1, t_0), (i_3, t_1)\}$ is blocked by $X'' = \{(i_1, t_1), (i_2, t_0)\}$.

3. Hard Bounds

Given a set of contracts Y, the choice correspondence of school s chooses a set of allocations that is contained in Y_s. To determine whether an allocation X' is in the choice set of Y, it is necessary to ensure no other allocation is preferred to it. This process cannot be carried out by considering contracts iteratively, as in the standard cumulative offer process. Example 3 shows that contracts that are removed at one stage may become acceptable to the school when a new contract is added. However, by reframing the problem to consider students with their set of available contracts, any student whose set of contracts is eliminated at one stage is not worth reconsidering.

Example 3. Let $I = \{i_1, i_2, i_3\}$, $S = \{s\}$, $T = \{t_0, t_1\}$. Let $i_1 \, \pi \, i_2 \, \pi \, i_3$. Assume $\bar{q}^s_{t_0} = \bar{q}^s_{t_1} = 1$. Assume $Y = \{(i_1, t_0), (i_3, t_1)\}$, $x = (i_1, t_1)$, and $x'' = (i_1, t_0)$. Then $\mathcal{C}^i(Y \cup \{x\}) = Y$ but $\mathcal{C}^i(Y \cup \{x\}) = \{x, x'\}$.

We consider a set of students $I' \subseteq i(Y_s)$ and their corresponding set of contracts $Z' = \{x \in Y_s : i(x) \in I'\}$. We determine whether the students in I' can be allocated to their schools using the contracts in Z'. To determine whether the students with contracts in Z' are in the choice function of school s, we look at the process of adding a student $i \notin i(Z')$. If it is possible to add an acceptable student i with contracts $Y_{i,s}$ to the school without replacing another student, then it is beneficial to add the student. Otherwise, the student has to replace student in Z'. This replacement is only beneficial if the student replaces a lower-quality student. Given this setup, a set of students is in the school's choice correspondence (i) if the seats can be allocated to the students, (ii) there are no acceptable students that can be added to the allocation without replacing another student, and (iii) if there are no feasible replacements that are leads to a preferred set of contracts.

3.1. Feasibility

Given a set of contracts $Z' \subseteq X$, we determine whether there is an allocation where all the students in $i(Z')$ are allocated to school s. To this end, we define the feasibility of Z' in a recursive manner. A set of contracts is feasible for a set of statuses T' for school s when (i) there are enough seats at school s to accommodate the students whose contracts are contained T' and (ii) there are enough seats to accommodate the students for any $T'' \subset T'$. A set of contracts is full for T' if it is feasible and it is impossible to add another student whose contracts are contained in T' without violating feasibility. Formally,

Definition 2. Let $Z' \subseteq X_s$ and let $T' \subseteq T$. A set of contracts Z' is

(i) **feasible** for T' if $|\{i \in i(Z') : t(Z'_i) \subseteq T''\}| \leq \sum_{t \in T''} \bar{q}^s_t$ for all $T'' \subseteq T'$.
(ii) **full** for T' if it is feasible for T' and $|\{i \in i(Z') : t(Z'_i) \subseteq T'\}| = \sum_{t \in T'} \bar{q}^s_t$.

We say that a set of contracts Z' is feasible if Z' is feasible for T.

When a set of contracts Z' is feasible for T', the students of $i(Z')$ whose contracts are contained in T' can be allocated to school s. It is necessary to examine the subsets $T'' \subseteq T'$ because students with contracts $t(Z'_i) \subseteq T''$ can only be allocated to seats with status T''. Therefore, when it is impossible to allocate these students to seats in T'', these students cannot be allocated to seats in T'. Further, we disregard students with $t(Z'_i) \nsubseteq T'$ because these students can be allocated to statuses outside T'. A feasible set of contracts Z' is full for T' when adding another student with contracts contained in T' violates feasibility. For a given Z', the set of full statuses determines whether a student can be added to to Z' without replacing another student. When a student needs to replace another student, the set of full statuses determines the students that can be replaced. The following example illustrates feasible and full sets of contracts.

Example 4. Let $I = \{i_1, i_2, i_3\}$, $S = \{s\}$, $T = \{t_0, t_1\}$, $\bar{q}_{t_0} = 2$ and $\bar{q}_{t_1} = 1$. When $Z' = \{(i_1, t_1), (i_2, t_0), (i_3, t_1), (i_3, t_0)\}$, Z' is full for $\{t_1\}$ and T, and Z' is feasible but not full for $\{t_0\}$. $X' = \{(i_1, t_1), (i_2, t_0), (i_3, t_0)\}$ assigns student in $i(Z')$ to seats in T. When $Z'' = \{(i_1, t_1), (i_2, t_0), (i_3, t_1)\}$, Z'' is not feasible for either $\{t_1\}$ and T. Here, $|\{i \in i(Z'') : t(Z''_i) \subseteq \{t_1\}\}| = 2$ but $\bar{q}_{t_1} = 1$. So, there is no allocation $X'' \subseteq Z''$ with $i(X'') = i(Z'')$.

In this example, the set Z' is full for T, so it is possible to allocate the students to seats. However, by removing an available contract, the new set of contracts Z'' violates feasibility, and it is impossible to assign the students to the seats.

These results generalize. When Z' is feasible for T', it is possible to allocate the students with $t(Z'_i) \subseteq T'$ to the school. However, when Z' is not feasible for T', the students with $t(Z'_i) \subseteq T'$ cannot be allocated to the school. Therefore, when Z' is feasible, it is possible to allocate all the students in $i(Z')$ to seats at the school. Otherwise, there is no allocation of all the students in $i(Z')$ to seats at the school. Formally,

Proposition 1. *$Z' \subseteq X_s$ is feasible for $T' \subseteq T$ if and only if there is an $X' \subseteq Z'$ where $i(X') = \{i \in i(Z') : t(Z'_i) \subseteq T'\}$.*

The proof of this and subsequent results are provided in Appendix A. Here, we provide a brief sketch of the proof. We use induction to show that the feasibility of a set of contracts implies the existence of allocations. When Z' is feasible for $T' = \{t\}$, students with $t(Z'_i) = \{t\}$ can be allocated to t. When the results hold for all feasible statuses of size up to size N, then for any T' of size $N + 1$ we can construct an allocation through an iterative process. Starting with Z', for each iteration we first check whether there is some $T'' \subset T'$ that is full. If not, we choose a student i with a Z_i that contains multiple contracts and remove all but one of their contracts from Z'. We repeat this process until either all the students have a single contract or there is a full $T'' \subset T'$. If all the students have a single contract, then this set of contracts is an allocation. When there is a full $T'' \subset T'$, through the inductive step we can assign the students with statuses contained in T'' to seats in T'' and the remaining students to seats in T'/T''. Conversely, when there is allocation $X' \subseteq Z'$ of all the students with $t(Z'_i) \subseteq T'$, the students with $t(Z'_i) \subseteq T''$ need to be allocated to seats contained in T'' for any $T'' \subseteq T'$. The number of students with contracts contained in T'' is at most $\sum_{t \in T''} \bar{q}^s_t$, so feasibility is satisfied.

3.2. Replaceability

Having characterized feasibility, we examine the process of adding student i to Z' when student i's contracts are not in Z'. When student i has a set of available contracts $Y_{i,s}$, student i can be added without replacing any students when $Z' \cup Y_{i,s}$ is feasible. Alternatively, when $Z' \cup Y_{i,s}$ is not feasible, student

i can replace student i' when $Z' \cup Y_{i,s}/Z'_{i'}$ is feasible. The minimum full set of contracts of Z' containing the statuses of $Y_{i,s}$ is critical for determining whether it is necessary to replace a student when adding i and which students i can replace. We define this below:

Definition 3. Assume $Z' \subseteq X_s$ is feasible. The statuses T' are **binding statuses** for $Y_{i,s}$ under Z' if

(i) $t(Y_{i,s}) \subseteq T'$

(ii) Z' is full for T'

(iii) There is no $T'' \subset T'$ such that $t(Y_{i,s}) \subseteq T''$ and Z' is full for T''

If there is no T' that is binding, we say that the binding set is unbounded.

If the binding set T' exists for $Y_{i,s}$ under Z', the **binding contracts** are $B(Y_{i,s}, Z') = \bigcup\{Z'_i : t(Z'_i) \subseteq T'\}$. The binding contracts are the contracts in Z' of students whose statuses are contained in T'. Using the binding contracts, the binding statuses are $t(B(Y_{i,s}, Z'))$ and the **binding students** are $i(B(Y_{i,s}, Z'))$. If there is no T' that is full and satisfies $t(Y_{i,s}) \subseteq T'$, then $Y_{i,s}$ is unbound under Z' and $B(Y_{i,s}, Z') = \infty$. In this situation the contracts in $Y_{i,s}$ are not constrained by Z'.[11]

When $B(Y_{i,s}, Z') = \infty$, the contracts of a student i can be added without replacing any students in $i(Z')$, so $Z' \cup Y_{i,s}$ is feasible. However, when $B(Y_{i,s}, Z') \neq \infty$, $Z' \cup Y_{i,s}$ is not feasible and student i cannot be added to the allocation without replacing another student. These properties are illustrated in Example 5.

Example 5. Let $I = \{i_1, i_2\}$, $S = \{s\}$, $T = \{t_0, t_1\}$. Assume $\bar{q}_{t_0} = \bar{q}_{t_1} = 1$. Let $Z' = \{(i_1, t_0)\}$, $Y_i = \{(i_2, t_0)\}$, and $Y'_i = \{(i_2, t_0), (i_2, t_1)\}$. Then Z' is full for only $\{t_0\}$, $B(Y_i, Z') = \{t_0\}$, and $B(Y'_i, Z') = \infty$. The allocation for Z' is $X' = \{(i_1, t_0)\}$. Here, $\{t_0\}$ and T are violated by $Z'' = Z' \cup Y_i$. $Z''' = Z' \cup Y'_i$ is feasible with the allocation $X'' = \{(i_1, t_0), (i_2, t_1)\}$.

When $B(Y_{i,s}, Z') \neq \infty$, it is possible to replace any student $i' \in B(Y_{i,s}, Z')$ by $i \in i(Y_{i,s})$. The new set of contracts $Z'' = Z' \cup Y_{i,s}/Z'_{i'}$ is feasible, and the students $i(Z'')$ can be allocated to the school. The binding statuses can

[11]The binding set of statuses and contracts are well-defined. We show this result in Appendix C.

contain statuses outside the students' status, allowing a student to replace another student even when their statuses don't intersect. Example 6 illustrates that the binding set depends on the set of contract offered through $Y_{i,s}$ and that it can be larger than the set of statuses in $t(Y_{i,s})$. It also shows that the contracts of other students can change to accommodate student i.

Example 6. Assume $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s\}$, $T = \{t_0, t_1\}$, $\bar{q}_{t_0} = 1$, and $\bar{q}_{t_1} = 2$. Let $Z' = \{(i_1, t_0), (i_2, t_1), (i_3, t_0), (i_3, t_1)\}$, $Y' = \{(i_4, t_0)\}$, and $Y'' = \{(i_4, t_1)\}$. Then $\{t_0\}$ and T are full, $B(Y', Z') = \{t_0\}$, and $B(Y'', Z') = B(Y''', Z') = T$. The allocation under Z' is $X' = \{(i_1, t_0), (i_2, t_1), (i_3, t_1)\}$. Under Y', i_4 can only replace i_1. Then $Z'' = Z' \cup Y'/Z'_{i_1}$ is feasible and has $X'' = X' \cup \{(i_4, t_0)\}/\{(i_1, t_0)\}$. Under Y'', i_4 can replace i_1, i_2 and i_3. Under $Z''' = Z' \cup Y''/Z'_{i_1}$ the allocation $X''' = \{(i_3, t_0), (i_2, t_1), (i_4, t_1)\}$ and i_3's admitted status changes from t_1 to t_0.

Examples 5 and 6 generalize. If student i has contracts $Y_{i,s}$ such that $B(Y_{i,s}, Z') = \infty$, she can be added to the allocation without needing to replace another student. However, if $B(Y_{i,s}, Z') \neq \infty$, she cannot be added to an allocation without replacing another student. Here, student i can replace a student $i' \in i(Z')$ if and only if the contracts of student i' are contained in $B(Y_{i,s}, Z')$. Therefore, the set of students that i can replace $i(B(Y_{i,s}, Z'))$.

Proposition 2. *Assume $Z' \subseteq X_s$ is feasible and let $i \notin i(Z')$.*

(i) $Z' \cup Y_{i,s}$ is feasible if and only if $B(Y_{i,s}, Z') = \infty$.
(ii) If $B(Y_{i,s}, Z') \neq \infty$, then $Z' \cup Y_{i,s}/Z'_{i'}$ is feasible if and only if $i' \in i(B(Y_{i,s}, Z'))$.

When $B(Y_{i,s}, Z') = \infty$, there is no T' such that Z' is full for T' and $t(Y_{i,s}) \subseteq T'$. Adding $Y_{i,s}$ to Z' increases the number of students whose contract is contained in T' by one; so T' becomes at most full. For any other T'', adding $Y_{i,s}$ to Z' does not change which students contracts are contained in T''. As a result, $Z' \cup Y_{i,s}$ is feasible. Alternatively, when $B(Y_{i,s}, Z') \neq \infty$, there is a T' such that Z' is full for T' and $t(Y_{i,s}) \subseteq T'$; therefore, the addition of i causes the feasibility of $Y_{i,s} \cup Z'$ to be violated by T'.

When student i with contract $Y_{i,s}$ replaces a student $i' \in i(B(Y_{i,s}, Z'))$, the only status that can violate feasibility are full sets T' that contains $t(Y_{i,s})$ but does not contain $t(Z'_{i'})$. The only T' possible T' satisfy $t(Y_{i,s}) \subseteq T'$ and $B(Y_{i,s}, Z') \nsubseteq T'$. However, Z' is not full for T'; otherwise, either T' is a binding set or there exists a $T'' \subseteq T'$ that is a binding set, a contraction of the uniqueness of the binding set. Therefore, $Z' \cup Y_{i,s}/Z'_{i'}$ is feasible.

Alternatively, for i to replace i', the contracts of student i' is are $B(Y_{i,s}, Z')$. Otherwise, if $Z'_{i'} \notin B(Y_{i,s}, Z')$, then $B(Y_{i,s}, Z')$ is full for Z' and i' is not in the binding set. Adding student i increases the number of student in the binding set by one, whereas removing student i' leaves the number of student in the binding set unchanged. As a result, the number of students in the binding set exceeds the number of available seats.

3.3. Choice Correspondence

We characterize the optimal replacement strategy for school s when schools consider students on a one-by-one basis. The school has a feasible set of contracts Z' and considers adding student i with contracts $Y_{i,s}$. When student i is acceptable and her contracts can be added to Z' without needing to replace another student, the school benefits from adding the student. When a student needs replace another student to be added, the replacement is only beneficial when the school prefers the new student to the student being replaced. When there is a beneficial replacement, the school's optimal replacement strategy is to replace the lowest ranked student it can. We call the lowest rank student that student i can replace the minimal admissible student:

Definition 4. Let $Z' \subseteq X_s$ be feasible, and let $i \in i(Y_s)/i(Z')$. The minimum admissible student is

$$i^*(Y_{i,s}, Z') = \begin{cases} \min_{\pi^s} i(B(Y_{i,s}, Z')) & \text{if } B(Y_{i,s}, Z') \neq \infty \\ \emptyset & \text{if } B(Y_{i,s}, Z') = \infty \end{cases}$$

The minimum admissible student is the lowest ranked student in Z' whose contracts are contained in $B(Y_{i,s}, Z')$. If the binding set is unbounded, there are some unoccupied seats and student i can be admitted without replacing another student. When the binding set is bounded, the lowest ranking student that i can replace is the minimal admissible student. The following result states that it is optimal to replace the minimal admissible student when the candidate student is preferred to the minimal admissible student.

Proposition 3. *Let $Z' \subseteq X_s$ be feasible with $i' \succ_s \emptyset$ for all $i' \in i(Z')$. Assume $i \in i(Y_s)/i(Z')$, and define $i^* = i^*(Y_{i,s}, Z')$. If $i' \succ_s i^*$ then there is* *an* $X'' \subseteq Z'' = Z' \cup Y_{i,s}/Z'_{i^*}$ *such that (i)* $i(X'') = i(Z'')$*, and (ii)* $X'' \succ_s X'$ *for any* $X' \subset Z' \cup Y_{i,s}$ *where* $i(X'') \neq i(X')$*.*

If X' does not fill the maximum number of seats, adding a contract from $Z_i \cup Y_{i,s}$ to X' creates an allocation that is preferred to X'; therefore, we consider X' that fill the maximum number of seats. For a maximal X', there is a $X''' \subseteq Z''$ that has the same contracts as X' for contracts that are not contained in a full set. By construction, X''' fills as many privileged seats as X'. The allocations only differ by a student; the minimum admissible student is in X' and a student that is preferred to the minimal admissible student is in X'''. Therefore, $X''' \succsim_s X'$. The allocation X'', that is equal to a maximal X''' under $\succsim_s$, is preferred to any other allocation in $Z' \cup Y_{i,s}$.

The minimum admissible student provides a concise way to determine whether a set of contracts is in the choice correspondence of school s.

Proposition 4. *Let* $Y \subseteq X$*.* $X' \in \mathcal{C}^s(Y)$ *for some* $X' \subseteq Z' \subseteq X_s$ *such that* $i(X') = i(Z')$ *if and only if* Z' *satisfies the following conditions:*

(i) Feasibility: Z' *is feasible.*

(ii) Acceptability: $i \, \pi^s \, \emptyset$ *for all* $i \in i(Z')$*.*

(iii) No beneficial replacement: $i^*(Y_{i,s}, Z') \succ_s i$ *for all* $i \in i(Y_s)/i(Z')$*.*

If X' is in the schools choice correspondence, then X' is an allocation; therefore, $Z' = \{x \in Y_{i,s} : i \in i(X')\}$ is feasible. There are no unacceptable students in X'; otherwise, the allocation containing only the contracts of the acceptable students of X' would be preferred to X'. For an allocation to be in the choice correspondence, there can be no student that is not in the allocation that is preferred to the minimal admissible student; otherwise, the school can do better by adding i student to its allocation, either without replacing a student if $i^* = \emptyset$ or by replacing i^* if $i^* \neq \emptyset$.

Alternatively, the poset $\succsim_s$ has a set of maximal elements under $\succsim_s$ which satisfy conditions (i), (ii), and (iii). Therefore, if the maximal Z' have the same students, they are equivalent and these allocations are preferred to every other allocation. On the contrary, if there were $Z'' \neq Z'''$ that satisfied conditions (i), (ii), and (iii), then there would be some maximal student in of the student only one of the two sets of contracts. This student could replace a lower ranked student in the other set of contracts, thereby contradicting the no beneficial replacement condition. The proof of this result implies that the choice correspondence is well defined and every allocation in the choice correspondence contains the same set of students.

To find the choice correspondence of school s, we follow an iterative process. Starting with $Z' = \emptyset$, we go through the school's available contracts on a student-by-student basis. If the student is preferred to the minimal admissible student, then that student's contracts are added to Z' and the minimal admissible student's contracts are removed.

Algorithm 1. Let $Y \subseteq X$ with $i(Y_s) = \{i_j\}_{j \in \{1,\ldots,|i(Y_s)|\}}$. Let $Z^{(0)} = \emptyset$. For each $j \in \{1, \ldots, |i(Y_s)|\}$, let $i^* = i^*(Y_{i_j,s}, Z^{(j-1)})$.

$$Z^{(j)} = \begin{cases} Z^{(j-1)} \cup Y_{i_j,s}/Y_{i^*,s} & \text{if } i_j \, \pi^s \, i^* \\ Z^{(j-1)} & \text{otherwise} \end{cases}$$

The final set of contracts is $Z' = Z^{(|i(Y_s)|)}$.

Proposition 5. *The set of contracts Z' chosen by Algorithm 1 is unique and satisfies $i(\mathcal{C}^s(Y)) = i(Z')$.*

The algorithm gives a feasible set of students that is contained in the school's choice correspondence. Students are considered on a one-by-one basis. The candidate student's contracts are added to the Z' when adding the student either increases the number of privileged seats or overall student quality. Both the number of privileged students and the overall student quality increases as more students are added to Z'; therefore, any student not in the final allocation is worse than the minimal admissible student. Therefore, the conditions of Proposition 4 are satisfied and the students in $i(Z')$ are in the schools choice correspondence.

4. Soft Bounds

To determine the school's choice correspondence under soft bounds, we consider a set of contracts Z' as we did under hard bounds. However, we partition Z' into students assigned to privileged seats and students assigned to general seats. Given this structure, we characterize feasibility, replaceability, and the choice correspondence accounting for the soft bounds.

Given a set of available contracts Y, the school chooses a set of students $I' \subseteq i(Y_s)$ with corresponding contracts $Z' = \{x \in Y_s : i(x) \in I'\}$. The students are partitioned into students assigned to privileged seats, $\tilde{I}'$, and students assigned to general seats, $\hat{I}'$. The available contracts for the students assigned to privileged and general seats are $\tilde{Z}' = \{x \in Y_s : i(x) \in \tilde{I}'\}$ and $\hat{Z}' = \{x \in Y_s : i(x) \in \hat{I}'\}$, respectively. For students in Z' to be allocated to their designated seats, we require $t(\tilde{Z}'_i) \cap T^p \neq \emptyset$ for all $i \in \tilde{I}'$ and $t_0 \in t(\hat{Z}'_i)$ for all $i \in \hat{I}'$.

4.1. Feasibility

We define feasibility similarly to under hard bounds, but account for the soft bounds. Feasibility is defined over sets of privileged students, $\tilde{Z}'$. It is also defined for partitions, Z', and accounts for the soft bounds by allowing any unfilled privileged seats to be available to general students.

Definition 5. Let $Z' \subseteq X_s$ and $T' \subseteq T^p$. A set of contracts Z' is

(i) **feasible** for T' if $|\{i \in i(\tilde{Z}') : t(\tilde{Z}'_i) \cap T^p \subseteq T''\}| \leq \sum_{t \in T''} \bar{q}^s_t$ for all $T'' \subseteq T'$.

(ii) **full** for T' if Z' is feasible for T' and $|\{i \in i(\tilde{Z}') : t(\tilde{Z}'_i) \cap T^p \subseteq T'\}| = \sum_{t \in T'} \bar{q}^s_t$

A set of contracts Z' is feasible if $\tilde{Z}'$ is feasible for T^p and $|i(Z')| \leq \bar{q}^s$.

Feasibility for soft bounds is similar to feasibility for hard bounds. However, students in $\tilde{Z}'$ are only allocated to privileged seats; therefore, we ignore their general statuses when considering the feasibility for $T \subseteq T^p$, Further, we account for the soft bounds by allowing any unfilled privileged seats to be filled by general students.

Just as for hard bounds, feasibility is linked to the existence of allocations under the school's available contracts. A set of contracts $\tilde{Z}'$ is feasible for T' if and only if the students in $\tilde{Z}'$ whose privileged contracts are contained in T' can be allocated to the privileged seats. A partition Z' is feasible if and only if there is an allocation where students in $\tilde{Z}'$ and $\hat{Z}'$ are allocated to privileged and general seats, respectively.

Proposition 6. *Let $Z' \subset X_x$.*

(i) *$\tilde{Z}'$ is feasible for $T' \subseteq T^p$ if and only if there is an $X' \subseteq \tilde{Z}'_{T^p}$ where $i(X') = \{i \in i(\tilde{Z}') : t(\tilde{Z}'_i) \cap T^p \subseteq T'\}$.*

(ii) *$Z' \subseteq Y$ is feasible if an only if there is an allocation $X' \subseteq Z'$ such that $i(X'_{T^p}) = i(\tilde{Z}')$ and $i(X') = i(Z')$.*

For a set of privileged status T', we consider the privileged contracts in $\tilde{Z}'$ and apply Proposition 1. Therefore, a set a contracts is feasible for T' if and only if the students in $\tilde{Z}'$ whose privileged contracts are contained in T' can be allocated to privileged seats. When Z' is feasible, $\tilde{Z}'$ is feasible for T^p, so there is an allocation of the students in $\tilde{Z}'$ to privileged seats. Because $|Z'| \leq \bar{q}^s$, the students in $\hat{Z}'$ can be allocated to general seats. Alternatively, if there is an allocation X' that allocates the students in $\tilde{Z}'$ privileged seats, then $\tilde{Z}'$ is feasible for T^p. Further, X' is an allocation, so $|i(Z')| = |X'| \leq \bar{q}^s$.

We define the binding set for privileged seats considering that students in $\tilde{Z}'$ are allocated to privileged seats. Just as in the definition of feasibility, we disregard the general contracts of students in $\tilde{Z}'$.

Definition 6. Assume $\tilde{Z}' \subseteq X_s$ is feasible for T^p. A set of statuses $T' \subseteq T^p$ is binding for the contracts $Y' \subseteq X_s$ under $\tilde{Z}'$ if

(i) $t(Y' \cap T^p) \subseteq T'$
(ii) $\tilde{Z}'$ is full for T'
(iii) There is no T'' such that $t(Y' \cap T^p) \subseteq T'' \subset T'$ and $\tilde{Z}'$ is full for T''

If there is no T' that is binding, we say that the binding set is empty.

The binding contracts, $B(Y', \tilde{Z}')$, are defined similarly to under hard bounds. If a binding set T' exists, then $B(Y', \tilde{Z}') = \bigcup\{\tilde{Z}'_i : t(\tilde{Z}'_i) \cap T^p \subseteq T'\}$. If, under soft bounds, Y' does not contain any privileged contacts, then $B(Y', \tilde{Z}') = \emptyset$. If there is no T' such that $\tilde{Z}'$ is full for T' with $t(Y' \cap T^p) \subseteq T'$, then $B(Y', \tilde{Z}') = \infty$. The binding students and binding statuses are $i(B(Y', \tilde{Z}'))$ and $t(B(Y', \tilde{Z}'))$, respectively.

For soft bounds, we consider Y' that can contain multiple students. This structure is useful for determining whether a partition maximizes the number of privileged students and whether a general student can replace a privileged student. When there are multiple students, the process of replacing students in $\tilde{Z}'$ is similar to when Y' has a single student. When the binding set is unbounded, it is possible to add some student from Y' to Z' and maintain feasibility. When the binding set is full, it is possible to replace a student in the binding set by some student in $i(Y')$. Formally,

Proposition 7. *Let $\tilde{Z}' \subset X_s$ be feasible. Let $Y' \subseteq X_s$ and $i(Y') \cap i(\tilde{Z}') = \emptyset$.*

(i) *$B(Y', \tilde{Z}') = \infty$ if and only if $\tilde{Z}' \cup Y'_i$ is feasible for T^p for some $i \in i(Y')$.*

(ii) If $B(Y', \tilde{Z}') \neq \infty$, then for any $i' \in i(B(Y', \tilde{Z}'))$, $\tilde{Z}' \cup Y'_i / Z'_{i',s}$ is feasible for T^p for some $i \in i(Y')$.

When $B(Y', \tilde{Z}') = \infty$, there is a student in $i(Y')$ whose contracts are not contained in a full set. Therefore, the student's contracts can be added to $\tilde{Z}'$ without needing to replace another student. Alternatively, if $\tilde{Z}' \cup Y'_i$ feasible for T^p, then when Y'_i is removed, any set of contracts containing $t(Y'_i)$ is not full under $\tilde{Z}'$. As any set containing Y' also contains Y'_i, no full sets of $\tilde{Z}'$ contain $t(Y')$. When $B(Y', \tilde{Z}') \neq \infty$ and student $i' \in B(Y', \tilde{Z}')$ is removed from $\tilde{Z}'$; there are no full sets containing Y'. As $B(Y', \tilde{Z}'/\tilde{Z}'_{i'}) = \infty$, some student in Y' can be added to the allocation, thereby replacing student i'.

We impose additional structure on the partition. A partition Z' is valid when it is feasible and there is no student in $i(\hat{Z}')$ whose contracts can be added to $\tilde{Z}'$ while maintaining feasibility; therefore, Z' is valid if it is impossible to fill more privileged seats using the students in $\hat{Z}'$

Definition 7. A partition $Z' = \tilde{Z}' \cup \hat{Z}'$ is **valid**, if Z' is feasible and there is no $i \in \hat{I}'$ such that $\tilde{Z}' \cup \hat{Z}'_i$ is feasible for T^p.

When a Z' is valid, it is impossible to add a student from $\hat{Z}'$ to $\tilde{Z}'$; therefore, $B(\hat{Z}', \tilde{Z}') \neq \infty$. Alternatively, if $B(\hat{Z}', \tilde{Z}') \neq \infty$, then $\hat{Z}'$ is contained in a full set. So, any $\hat{Z}'_i$ is also contained in a full set and cannot be added to $\tilde{Z}'$ with replacing another student. This proves the following corollary:

Corollary 1. *A feasible partition Z' is valid if and only if $B(\hat{Z}', \tilde{Z}') \neq \infty$.*

We analyze valid partitions because they fill the maximum number of privileged seats given the set of contracts in Z'; therefore, schools always prefer a valid partition to other partitions containing the same contracts. Further, valid partitions with the same set of students fill the same number of privileged seats; therefore, the school is indifferent between them. These results, stated formally below, allow us to restrict our attention to valid sets of contracts.

Proposition 8. *Let $Z' \subseteq X_s$ be feasible. Then*

(i) There exists a valid partition of $Z'' = Z'$ such that $|i(\tilde{Z}'')| \geq |i(\tilde{Z}')|$.
(ii) Any valid partitions Z' and Z'' with $Z' = Z''$ have $|i(\tilde{Z}')| = |i(\tilde{Z}'')|$.

For a partition Z', we construct a valid partition by considering the students in $i(\hat{Z}')$ on a one-by-one basis and iteratively adding their contracts to $\tilde{Z}'$ when $B(\hat{Y}_{i,s}, \tilde{Z}') = \infty$. As more students are added to $\tilde{Z}'$, set of statuses that becomes full will continue to be full. Students that are not added to $\tilde{Z}'$ cannot be added $\tilde{Z}'$ at the end of the process, so the final allocation is valid. Valid partitions with the same contracts have the same number of privileged students because the union of statuses that are full is the same and the set of students not contained in full statuses is the same. Therefore, the number of privileged students must be equal.

As valid partitions with the same contracts have the same students and fill the same number of privileged seats, schools will be indifferent between the allocations associated them. Thus, the preferences over the allocations induce preferences over Z'. Specifically, for valid Z' and Z'' and their associated X' and X'', we say $Z' \succsim_s Z''$ if and only if $X' \succsim_s X''$.

4.2. Replaceability

We consider whether it is beneficial to add student $i \notin i(Z')$, with contracts $Y_{i,s}$, to Z'. We determine when the contracts of student i can be added to Z' without replacing another student and when it is necessary to replace a student in Z'. When it is necessary to replace a student, there are multiple ways student i can replace student $i' \in i(Z')$. The student can directly replace i' without affecting other students in the partition, or i can indirectly replace i' with some other students transferring between privileged and general seats. The span of a set of contracts determines replaceability.

Definition 8. Let $Y' \subseteq X_s$ and let $Z' \subseteq X_s$ be valid. Then

$$Span(Y', Z') = \begin{cases} \hat{B}(Z') \cup B(\hat{Z}' \cup Y', \tilde{Z}') & \text{if } t_0 \in t(Y' \cup B(Y', Z')) \\ B(Y', \tilde{Z}') & \text{otherwise} \end{cases}$$

where $\hat{B}(Z') = \hat{Z}'$ if $|Z'| = \bar{q}^s$ and $\hat{B}(Z') = \infty$ if $|Z'| < \bar{q}^s$.

The $Span(Y_{i,s}, Z')$ is the set of contracts that student i can replace. It allows for direct replacement, when a student replaces a student in the binding set, and indirect replacement, through chains of replacement across partitions. For instance, student i can replace a student in $B(Y_{i,s}, \tilde{Z}')$, who can replace a student in the general partition, who in turn can replace a privileged student i' who is not in $B(Y_{i,s}, \tilde{Z}')$. Under soft bounds, the minimal admissible student is the lowest-ranked student whose contracts are contained in $Span(Y', Z')$.

Definition 9. Let $Y' \subseteq X_s$ and let $Z' \subseteq X_s$ be valid. The minimum admissible student is

$$i^*(Y', Z') = \begin{cases} \min_{\pi^s} i(Span(Y', Z')) & \text{if } Span(Y', Z') \neq \infty \\ \emptyset & \text{if } Span(Y', Z') = \infty \end{cases}$$

The minimal admissible student helps to establish the optimal replacement procedure. When i is acceptable and $i^*(Y_{i,s}, Z') = \emptyset$, adding student i fills more privileged seats or increases student quality. Therefore, student i should be added. When $|i(Z')| < \bar{q}^s$, student i can be added without removing another student. However, when $|i(Z')| = \bar{q}^s$, it is necessary to remove a student from Z'. Removing the minimum ranked student in the span of $\hat{Z}'$ is optimal. Formally,

Proposition 9. *Assume $Z' \subseteq X_s$ is valid, $Y_{i,s} \neq \emptyset$ for some $i \notin i(\tilde{Z}')$, $i \, \pi^s \, \emptyset$ for all $i \in i(Z' \cup Y_{i,s})$, and $i^*(Y_{i,s}, Z') = \emptyset$.*

(i) If $|Z'| < \bar{q}^s$ then $Z'' = Z' \cup Y_{i,s}$ is valid.

(ii) If $|Z'| = \bar{q}^s$ then $Z'' = Z' \cup Y_{i,s}/Z'_{i'}$ is a valid where $i' = i^*(\hat{Z}', Z')$.

Further, $Z'' \succ_s Z'''$ for any valid $Z''' \subseteq Z' \cup Y_{i,s}$ where $Z''' \neq Z''$.

When $i^*(Y_{i,s}, Z') = \emptyset$ and $|Z'| < \bar{q}^s$, student i can be added without replacing another student. To maintain validity, i must be added to $\tilde{Z}'$ when $B(Y_{i,s}, \tilde{Z}') = \infty$. However, when $B(Y_{i,s}, \tilde{Z}') \neq \infty$, either student i is added to $\hat{Z}'$ or student i is added to $\tilde{Z}'$ and a student in $B(Y_{i,s}, \tilde{Z}')$ is added to $\hat{Z}'$. Z'' is preferred to other possible allocations, because the other allocations contain a subset of the students in Z''; therefore, they fill fewer privileged seats and have lower student quality.

When $i^*(Y_{i,s}, Z') = \emptyset$ and $|Z'| = \bar{q}^s$, there are unoccupied privileged seats that are used for general students. To add student i to $\tilde{Z}'$, it is necessary to remove a student from the general seats. This is achieved by either removing a student from $\hat{Z}'$ or by removing a student from $B(\hat{Z}', Z')$ and replacing the student with a student from $\hat{Z}'$. By removing $i^*(\hat{Z}', Z')$ and reallocating a student from $\hat{Z}'$ to $\tilde{Z}'$ as necessary, student quality is maximized compared to the other replacements that fill the same number of privileged seats. Further, the school prefers the replacements to any Z''' with fewer students; therefore, Z'' is the optimal replacement.

When $i^* = i^*(Y_{i,s}, Z') \neq \emptyset$, student i cannot be added to the allocation without removing a student from Z'. The set of students that i can replace are the students in $Span(Y_{i,s}, Z')$; therefore, when $i \succ i^*$, the school wants to replace the minimal admissible student in this set. Student i will be added to Z' and student i^* will be removed.

Proposition 10. *Assume Z' is valid, $Y_{i,s} \neq \emptyset$ for some $i \notin i(\tilde{Z}')$, and $i^* = i^*(Y_{i,s}, Z')\, \pi^s\, \emptyset$. Then $Z'' = Z' \cup Y_{i,s}/Z_{i^*}$ is a valid. Further, if $i\, \pi^s\, i^*$ then $Z'' \succ_s Z'''$ for any valid $Z''' \subseteq Z' \cup Y_{i,s}$ where $Z''' \neq Z''$.*

To construct a valid Z'', we use a chain of replacements that starts with student i being added and ends with student i^* being removed. At each stage in the chain, the student that was previously removed is added and another student is removed. Through this process, student i can replace any student in the $Span(Y_{i,s}, Z')$. More specifically, if i^* is in $B(Y_{i,s}, \tilde{Z}')$ then a single replacement is necessary. When $i^* \in i(\hat{Z}')$, either $t_0 \in t(Y_{i,s})$ and i can directly replace i^*, or there is a chain where i replaces i' for some i' in $B(Y_{i,s}, \tilde{Z}')$ and i' replaces i^*. Through this chain, i' moves from $\tilde{Z}'$ to $\hat{Z}'$. This process can be extended further. If i^* is in $B(\hat{Z}', \tilde{Z}')$, there is some $i'' \in \hat{Z}'$ that can replace i^*. Student i'' can be replaced by i either directly or indirectly, through some i' in $B(Y_{i,s}, \tilde{Z}')$.

Z'' is preferred to any other allocation contained in $Y_{i,s} \cup Z'$. The replacement removes the minimal ranked student, and fills the same number of privileged seats. Replacing a different student in the span will lead to lower student quality, whereas replacing a student who is not in the span will reduce the number of privileged students. The school prefers the replacements to allocations with fewer students, so Z'' is the optimal replacement.

4.3. Choice Correspondence

Propositions 9 and 10 are useful for characterizing the school's choice correspondence. They imply that whenever student i is preferred to the minimum admissible student there is an allocation including student i that is preferred to Z'. Therefore, if we restrict our attention to valid partitions, an allocation is in the school's choice correspondence if all the students in Z' are acceptable and there are no students in Y_s/Z' and are preferred to the minimum admissible student.

Proposition 11. *Let $Y \subseteq X$. $X' \in \mathcal{C}^s(Y)$ if and only if the Z' where $\tilde{Z}' = Y_{i(X'_{Tp}),s}$ and $\hat{Z}' = Y_{i(X'_{t_0}),s}$ satisfies the following conditions:*

(i) Validity: Z' is a valid partition

(ii) Acceptability: $i \, \pi^s \, \emptyset$ for all $i \in i(Z')$.

(iii) No beneficial replacement: $i^*(Y_{i,s}, Z') \, \pi^s \, i$ for all $i \in i(Y_s)/i(Z')$

This proposition is analogous to Proposition 4, and follows using similar reasoning. When Z' is the set of contracts associated with an $X' \in C^s(Y)$, Z' has similar attributes to X'. As X' is an optimal allocation, Z' must be valid. Removing unacceptable students from X' would increase student quality, so there are no unacceptable students in Z'. Finally, there are no students in Y_s/Z' that are preferred minimum admissible student; otherwise, adding the student to Z' is beneficial and the associated allocation is preferred to X'. Alternatively, the Z' associated with an allocation in the choice set satisfies conditions (i), (ii), and (iii). As a unique Z' satisfies these conditions, the associated X' is in the schools choice correspondence.

Similarly to under hard bounds, the school's choice correspondence can be determined using an iterative process. We consider the school's acceptable contracts on a student-by-student basis. When $i^* = \emptyset$, student i is accepted. If the soft-bound seats are full, a student in Z' is removed to accommodate the new student. When $i^* \neq \emptyset$, the student is accepted if she is preferred to the minimal admissible student, and the minimal admissible students is removed. The movement of students between $\tilde{Z}'$ and $\hat{Z}'$ is specified in the proofs of Propositions 9 and 10.

Algorithm 2. Let Y with $i(Y) = \{i_j\}_{j \in \{1,\ldots,|i(Y)|\}}$ such that $i_j \, \pi^s \, \emptyset$ for all j. Start with $Z^0 = \emptyset$.

For each student, i_j, determine $i^* = i^*(Y_{i_j,s}, Z^{j-1})$.

1. If $i^* = \emptyset$ and $|i(Z^{j-1})| < q^s$, let $\tilde{Z}^j = \tilde{Z}^{j-1} \cup Y_{i_j,s}$
2. If $i^* = \emptyset$ and $|i(Z^{j-1})| = q^s$, let $\tilde{Z}^j = \tilde{Z}^{j-1} \cup Y_{i_j,s}/\tilde{Z}^{j-1}_{i'}$ where $i' = i^*(\hat{Z}^{j-1}, \tilde{Z}^{j-1})$
3. If $i^* \succsim_s \emptyset$ and $i_j \, \pi^s \, i^* = i^*(Y_{i_j,s})$, let $Z^j = Z^{j-1} \cup Y_{i,s}/Z^{j-1}_{i^*}$

The following result shows that the algorithm chooses the schools in the students choice function.

Proposition 12. *The set of contracts Z' chosen by Algorithm 2 is unique and satisfies $i(\mathcal{C}^s(Y)) = i(Z')$.*

The algorithm considers students on a one-by-one basis. The candidate student's contracts are added when it increases the number of privileged seats or overall student quality. When necessary, an optimal replacement procedure is used to remove a student. The final Z' is preferred because any student not in the allocation was either rejected or removed at some stage. Adding the student would have lead to an inferior allocation at that stage. As the minimal admissible student is increasing as the algorithm progresses, adding the student to the final set of students would either decrease the number of privileged students or would leave the number of privileged students unchanged while decreasing student quality. Therefore, the conditions of Proposition 11 are satisfied and the students in $i(Z')$ are in the schools choice correspondence.

5. Properties

We look at the properties of the schools' choice correspondences by expanding the definitions of *substitutability*, *unilateral substitutability*, *bilateral substitutability*, and *irrelevance of rejected contracts* to account for the schools' weak preferences over allocations. These definitions coincide with their definitions for choice functions when the choice correspondences consist of a single set of contracts.

Definition 10.

A choice correspondence $\mathcal{C}^s$ satisfies

(i) **substitutability** when for all $x, x' \in X$ and $Y \subset X$ if $x \notin X'$ for any $X' \in \mathcal{C}^i(Y \cup \{x\})$ then $x \notin X'$ for any $X' \in \mathcal{C}^i(Y \cup \{x, x'\})$.

(ii) **unilateral substitutability (ULS)** when for all $x, x' \in X$ and $Y \subset X$ such that $i(x) \notin i(Y)$, if $x \notin X'$ for any $X' \in \mathcal{C}^s(Y \cup \{x\})$ then $x \notin X'$ for any $X' \in \mathcal{C}^s(Y \cup \{x, x'\})$.

(iii) **bilateral substitutability (BLS)** when for all $x, x' \in X$ and $Y \subset X$ such that $i(x), i(x') \notin i(Y)$, if $x \notin X'$ for any $X' \in \mathcal{C}^s(Y \cup \{x\})$ then $x \notin X'$ for any $X' \in \mathcal{C}^s(Y \cup \{x, x'\})$.

These conditions are successively weaker conditions that can be combined with IRC to ensure the existence of a stable allocation when students have strict preferences over their admitted status. Our choice correspondence violates substitutability and ULS, but satisfies BLS.

Proposition 13. *The school's choice correspondence correspondence, $\mathcal{C}^s$, violates substitutability and ULS, but satisfies BLS.*

Example 7, below, violates ULS; therefore, it violates substitutability. The example violates the conditions under both hard and soft bounds. The conditions are violated because the addition of a contract changes the schools preferred assignment of a student's status. Alternatively, the choice correspondence algorithms show that BLS is satisfied for the choice function under both soft bounds and hard bounds. Any student that is rejected at any stage under their contracts will not be acceptable when new students are considered, as the student is not preferred to the minimal acceptable student.

Example 7. Let $I = \{i_1, i_2, i_3\}$, $S = \{s\}$, $T = \{t_0, t_1\}$. Let $i_1 \, \pi \, i_2 \, \pi \, i_3$. Assume $\bar{q}^s = 2$ and $\bar{q}^s_{t_0} = \bar{q}^s_{t_1} = 1$. Let $Y = \{(i_1, t_0), (i_3, t_1)\}$, $x = (i_1, t_1)$, and $x' = (i_1, t_0)$. Then $\mathcal{C}^i(Y \cup \{x\}) = Y$ but $\mathcal{C}^i(Y \cup \{x\}) = \{x, x'\}$.

Definition 11. A choice correspondence $\mathcal{C}$ satisfies **irrelevance of rejected contracts (IRC)** when for all $Y \subset X$ and $x \in X/Y$, if $x \notin X'$ for all $X' \in \mathcal{C}(Y \cup \{x\})$ then $\mathcal{C}(Y) = \mathcal{C}(Y \cup \{x\})$.

A choice correspondence satisfies IRC when the set of allocations in the schools choice correspondence is unaffected by the addition of a contract that is not in any allocation of the schools choice correspondence. Aygün and Sönmez (2013) show that this condition is necessary to ensure that a stable allocation exists when students have strict preferences over their admitted status.

Proposition 14. *The choice correspondence $\mathcal{C}^s$ satisfies IRC.*

For $\mathcal{C}^s$ the set of chosen allocations depend on the pairwise comparison of allocations using $\succsim_s$. The addition of an irrelevant contract will not affect the set of preferred allocations because allocations in the original choice set will remain preferred to the allocations that they were originally preferred to. The following result shows that there exists a stable allocation.

Proposition 15. *There exists a stable allocation.*

The results from Hatfield and Kojima (2010) and Aygün et al. (2012) show that when IRC and BLS are satisfied for C^s, there exists a stable allocation. However, this result does not directly apply to our framework.[12] Instead, we create quasi-contracts that include the contracts that students are indifferent between as one contract.[13] The schools' choice function includes a quasi-contract if one of the contracts in the quasi-contract is in the choice correspondence. Under this structure, the quasi-contracts satisfy BLS and IRC; therefore, there is a stable allocation of quasi-contracts. This allocation of quasi-contracts corresponds to an allocation of contracts.

6. Adaptive Assignment Algorithm

To accommodate the choice correspondences of the students and schools, we modify the student optimal cumulative offer process. We allow students to submit rankings of schools that include sets of contracts they are indifferent between. When a student has a contract that is in the school's choice correspondence, the school retains all the student's contracts. A contract that is not in the school's choice correspondence may be in the choice correspondence when a new set of contracts is added; therefore, this structure allows the set of contracts to adapt to the needs of the school. When none of a student's contracts are in the school's choice correspondence, those contracts are never acceptable to the school and can be rejected.

Under the adaptive assignment algorithm, each student applies to their highest ranked school under their highest ranked set of contracts. Given the set of students that apply, schools conditionally accept all the students in their choice correspondence and reject the remaining students. The rejected students then apply to another school with their next highest set of contracts, and the process repeats until every rejected student has no remaining contracts. Through this process, students can apply to a school multiple times when one set of contracts associated with the school is preferred to another.

We assume that students submit a preference list of the sets of statuses they are indifferent between. Formally, students i's preference list is found iteratively by defining $Y_i^1 = \mathcal{C}^i(X_i)$ and $Y_i^k = \mathcal{C}^i(X_i'/\bigcup_{l=1}^{k-1} Y_i^l)$. The student's

[12]Ties cannot be broken arbitrarily, as blocking coalitions can include students who are indifferent between their contracts under original allocation and the blocking allocation.

[13]We call these quasi-contracts because they depend on the student's preferences.

preference list is finite, each set of contracts includes a single school, and the sets of contracts is disjoint.

Algorithm 3. Students list their preferences, $\{Y_i^1, Y_i^2, Y_i^3, \dots\}_{i \in I}$

(i) Students offer $Y = \bigcup_{i \in I} Y_i^1$
(ii) Student i is conditionally accepted if $i \in i(\mathcal{C}^s(Y_i))$
(iii) Student i is rejected if $i \notin i(\mathcal{C}^s(Y_i))$
(iv) Keep the contracts Y_i^k in Y if student i is conditionally accepted
(v) Otherwise reject Y_i^k and add Y_i^{k+1} to Y and go to step (ii).

The algorithm terminates when there are no new contracts added to Y.

Given a final set of contracts Y, the students assigned to school s are $i(Y_s)$. The algorithm can choose any X' such that $X_i' \in \mathcal{C}^s(Y)$, as the students and school are indifferent between any of these allocations. Students are indifferent between their contracts in Y_i^k and schools are indifferent between any allocations in their choice correspondence.

The adaptive assignment algorithm has similar properties to the student proposing cumulative offer process in the standard school choice environment. The algorithm produces the student optimal stable matching (Gale and Shapley, 1962) and gives students the incentive to truthfully report their preferences (Roth, 1982). However, schools have similar strategic incentives as under the student proposing cumulative offer process. Schools may have incentives to strategically misreport their preferences over students (Roth, 1982) or reduce their quotas (Sönmez, 1997).

Proposition 16. *Given truthful reporting of preferences, the adaptive assignment algorithm chooses the student optimal stable allocation.*

The adaptive assignment algorithm chooses a stable allocation because each school weakly prefers the chosen allocation to any other allocation consisting of the students already considered by the school. Further, when an allocation includes a contract that has not been considered by the school, the students with a new contract are worse off under this contract than they are under the adaptive assignment algorithm outcome.

7. Conclusion

We develop a framework for analyzing a school choice problem where a student can simultaneously hold multiple privileged statuses. Our structure allows students to have weak preferences over their admitted status at a school but requires they have strict preferences over allocations that contain different schools. The school's primary objective is to fill as many privileged seats as possible. After that, it maximizes student quality. As many allocations fill the same number of privileged seats and have the same students, schools also have weak preferences over allocations.

We deal with weak preferences by allowing students to apply to schools with preference lists that contain the of sets of contracts that they are indifferent between. A school that finds at least one of a student's contracts acceptable holds all that student's contracts, giving it the flexibility to reassign the student to a different status when another new student is added to the school's choice set. Using this, we can use the school's choice correspondence over sets of contracts to determine the student-optimal stable allocation using a cumulative offer process.

We build on the previous literature, generalizing the overlapping quota model of Hatfield and Milgrom (2005) to allow students to have weak preferences over their admitted statuses. Additionally, our procedure provides a computationally efficient alternative to the method suggested in Hatfield and Milgrom (2005). To determine the school's choice set, Hatfield and Milgrom (2005) needs to consider all possible combinations of allocations of students to statuses. This procedure grows exponentially as the number of students a school considers increases. Alternatively, the processes in the adaptive assignment algorithm are linear. The minimal admissible student is found by choosing the lowest-ranked student in the span of the student's contracts.[14] The span can be determined by counting the number of students contained in each set of statuses.[15]

We also generalize the preferences of Aygün and Turhan (2017a) by allowing students to be indifferent to their admitted statuses. Beyond producing a stable allocation when students have weak preferences, our framework has

[14]For hard bounds, the minimal admissible student is the lowest-ranked student in the binding set.

[15]Computational efficiency can be improved by tallying the students in each set of status and updating the tally each time a new student is added to the allocation.

several advantages over Aygün and Turhan (2017a) and one drawback. From a practical standpoint, it may be easier for students who are indifferent between different contracts to list preferences over sets of contract rather than individual contracts. Further, our framework is symmetric in its treatment of different statuses, something that is impossible in deterministic capacity transfer mechanisms. Therefore, policymakers do not need to make subjective decisions that could arbitrarily influence the allocation.

However, our structure lacks the flexibility to reassign unused capacity that exists under the monotone capacity transfer in Aygün and Turhan (2017a). Their mechanism allows unused seats to be assigned to specific privileged students, allowing for the transfer of unused seats between similar types of statuses. Our structure can be modified to give other privileged students priority over unused seats; however, this complicates the structure and decreases computational efficiency. Therefore, if the number of seats that are not filled by privileged students is small, the benefits provided by the more robust reassignment procedure would be small.

Our mechanism differs from alternative mechanisms for constrained matching. In particular, it differs from slot specific priorities (Kominers and Sönmez, 2016). This model has multiple seat types and changes the priority rankings of students for the seats based on the students' characteristics. Each seat's priority ranking is fixed. In our framework, the acceptability of a student for a specific status changes depending on the statuses of the other students; therefore, seats do not have priority ranks. Our mechanism also differs from stable improvement cycles Erdil and Ergin (2008). When schools use random tie-breaking rules to allocate seats to students, the resulting allocation can be inefficient. Stable improvement cycles improve on these inefficient allocations by iteratively finding stable Pareto improvements. Through the process, they achieve the student-optimal stable matching. In our environment, the standard cumulative offer process can produce allocations that are Pareto optimal but not stable; therefore, there are no Pareto improvements that produce a stable allocation.

There are limitations in the matching with disjoint quotas models that have been pointed out in other work. As matching with disjoint quotas is a special case of matching with overlapping quotas, the limitations will apply to our work. In particular, quotas can make some minority students worse off **??** and do not maximize diversity across all schools (Bo, 2016). Rather, our structure gives privileged students priority until the quota threshold is met.

To implement our mechanism, two important market design features should be incorporated. The selection procedure should be structured to make it easy for students to state indifferences, such as allowing an option of applying to the school under all available statuses. Additionally, students must rank privileged statuses over the general status. This restriction ensures that the algorithm does not favor privileged students when equality is achieved.

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Appendix A. Proofs

Proof (Prop. 1). (⇒) We use induction. Let Z' is feasible for $T' = \{t'\}$. Define $i_{\subseteq}(Y, T') = \{i' \in i(Y) : t(Y) \subseteq T'\}$. Then $X' = \{(i, s, t') : t(Z'_i) \in T'\}$ has $i(X') = i_{\subseteq}(Z'_{i'}, T')$. Let (i,⇒) hold for all $T' \subseteq T$ such that Z' is feasible for T' and $|T'| \leq N$. Assume T' where Z' is feasible for T' and $|T'| = N + 1$.

We show (i,⇒). If there is a $T'' \subset T'$ such that Z' is full for T'', by the inductive step there is an X'' with $i(X'') = i_{\subseteq}(Z'_i, T'')$. Define $T''' = T'/T''$. We show $Z'' = \{x \in Z' : t(x) \not\subseteq T''\}$ is feasible for T'''. Let $T^{(4)} \subseteq T'''$. Then $T^{(4)} \cup T'' \subseteq T'$ so Z' is feasible for $T^{(4)} \cup T'$. Further, $x' \in Z''$ *iff* $x' \notin \{x \in Z' : Z'_i \subseteq T''\}$; therefore,

$$\begin{aligned} |i_{\subseteq}(Z'', T^{(4)})| \leq & |i_{\subseteq}(Z', T^{(4)} \cup T'')| - |i_{\subseteq}(Z', T'')| \\ \leq & \sum_{t \in T'} \bar{q}^s_t - \sum_{t \in T''} \bar{q}^s_t = \sum_{t \in T'^{(4)}} \bar{q}^s_t \end{aligned}$$

So Z'' is feasible for T''' and $|T'''| \leq N$; therefore, there is an X''' such that $i(X''') = i_{\subseteq}(Z'', T^{(4)})$. Define $X' = X'' \cup X'''$. If $i' \in i(X')$, then $i' \in i(X'')$ or $i' \in i(X''')$. In either case $i' \in i_{\subseteq}(Z', T')$. Alternatively, let $i' \in i_{\subseteq}(Z', T')$. If $t(Z'_i) \subseteq T''$ then $i' \in i(X'') \subseteq i(X')$. If $t(Z'_{i'}) \not\subseteq T''$ then $i \in i(X''') \subseteq i(X')$. So, $i(X') = i_{\subseteq}(Z', T')$.

If no $T'' \subset T'$ is full for Z'. Order $I' = i_{\subseteq}(Z', T')$ by $I' = \{i_1, \ldots, i_{|I'|}\}$. Let $Z^0 = \bigcup_{t(Z_i) \subseteq T'} Z'_i$, define Z^j iteratively: for $j = 1, \ldots, |I'|$ choose an $x \in Z'_{i_j,s}$ and set $Z^j = (Z^{j-1}/Z'_{i_j}) \cup x$. As Z' is not full for any $T'' \subset T'$, for $j = 0$, $|i_{\subseteq}(Z', T'')| < \sum_{t \in T'} \bar{q}^s_t$ for all $T'' \subset T'$. If $|i_{\subseteq}(Z', T'')| < \sum_{t \in T'} \bar{q}^s_t$ for all j and all $T'' \subset T'$, then $X' = Z^{|I'|}$ has $|X'| = 1$ for $i' \in I'$, and satisfies the $|X'_t| < \bar{q}^s_t$ for all $t \in T$; therefore, X' is an allocation. By construction, $i(X') = i_{\subseteq}(Z', T')$. If $|i_{\subseteq}(Z', T'')| \geq \sum_{t \in T'} \bar{q}^s_t$ for some j and $T'' \subset T'$, take j^* as the minimum j for which it fails, and let $T''' \subset T'$ be a set for which it fails at $j = j^*$. Comparing $j^* - 1$ to j^*, the only i for which $t(Z^j_i)$ changes is

i_{j^*}; therefore, $|i_\subseteq(Z', T''')| = \sum_{t\in T'} \bar{q}^s_t$ and $T''' \subset T'$ is full. By the early part of the proof, there is an X' such that $i(X') = i_\subseteq(Z', T')$.
($\Leftarrow$) Let $T' \subseteq T$. Assume $X' \subseteq Z'$ satisfies $i(X') = i_\subseteq(Z', T')$. For any $T'' \subseteq T'$, $\sum_{t\in T''} X'_t \leq \sum_{t\in T''} \bar{q}^s_t$. Further, $t(Z'_i) \subseteq T''$ implies $t(X'_i) \subseteq T''$; therefore, $|i_\subseteq(Z', T'')| \leq |X'_{T''}| \leq \sum_{t\in T''} \bar{q}^s_t$. So, Z'_i is feasible for T'.

Proof (Prop. 2). **(i)** Let $B(Y_{i,s}, Z') = \infty$. Define $Z'' = Z' \cup Y_{i,s}$. Define $i_\subseteq(Y, T') = \{i' \in i(Y) : t(Y) \subseteq T'\}$. As Z' is feasible, $|i_\subseteq(Z', T')| \leq \sum_{t\in T'} \bar{q}^s_t$ for all $T' \subseteq T$. For any T' where $t(Y_{i,s}) \not\subseteq T'$, $i_\subseteq(Z'', T') = i_\subseteq(Z', T')$; therefore, $|i_\subseteq(Z'', T')| \leq \sum_{t\in T'} \bar{q}^s_t$. As $B(Y_{i,s}, Z') = \infty$, $|i_\subseteq(Z', T')| < \sum_{t\in T'} \bar{q}^s_t$ when $t(Y_{i,s}) \subseteq T'$. As $i_\subseteq(Z'', T') \subseteq i_\subseteq(Z', T') \cup \{i\}$, $|i_\subseteq(Z'', T')| \subseteq |i_\subseteq(Z', T')| + 1 \leq \sum_{t\in T'} \bar{q}^s_t$. Therefore, Z'' is feasible. Alternatively, let $t(B(Y_{i,s}, Z')) = T'$. Then $|i_\subseteq(Z', T')| = \sum_{t\in T'} \bar{q}^s_t$. Because $t(Y_{i',s}) \subseteq T'$, $|i_\subseteq(Z'', T')| = |i_\subseteq(Z', T')| + 1 > \sum_{t\in T'} \bar{q}^s_t$ and $Y_{i,s} \cup Z'$ is not feasible.

(ii) Define $Z'' = Z' \cup Y/Z'_{i''}$. Then $|i_\subseteq(Z'', T'')| \leq |i_\subseteq(Z', T'')| + 1$ as $i_\subseteq(Z'', T'')/i_\subseteq(Z', T'') \subseteq \{i'\}$. Therefore, any T'' with $|i_\subseteq(Z', T'')| < \sum_{t\in T''} \bar{q}^s_t$ satisfies $|i_\subseteq(Z'', T'')| \leq |i_\subseteq(Z', T'') + 1| \leq \sum_{t\in T''} \bar{q}^s_t$. Therefore, we consider T'' where Z' is full for T'' and: (a) $T' \subseteq T''$, (b) $t(Y) \not\subseteq T''$, or (c) $t(Y) \subseteq T''$ and $T' \not\subseteq T''$.
(a) $T' \subseteq T''$: Then $i' \notin i_\subseteq(Z', T'')$, $i' \in i_\subseteq(Z'', T'')$, $i'' \in i_\subseteq(Z', T'')$, and $i'' \notin i_\subseteq(Z'', T'')$; therefore, $|i_\subseteq(Z'', T'')| = |I' \cup \{i'\}/\{i''\}| = |i_\subseteq(Z', T'')|$. So $|i_\subseteq(Z'', T'')| = |i_\subseteq(Z', T'')| = \sum_{t\in T''} \bar{q}^s_t$. **(b)** $t(Y) \not\subseteq T''$: As $i' \notin i_\subseteq(Z', T'')$,$i \notin i_\subseteq(Z'', T'')$, and $i'' \notin i_\subseteq(Z'', T'')$, $|i_\subseteq(Z', T'')| = |I' \cup \{i'\}/\{i''\}| \leq |i_\subseteq(Z', T'')| = \sum_{t\in T''} \bar{q}^s_t$. **(c)** $t(Y) \subseteq T''$ and $T' \not\subseteq T''$: If $T'' \subset T'$ this contradicts $T' = t(B(Y_{i,s}, Z'))$. If $T'' \not\subset T'$, then either T'' is binding or some $T''' \subset T''$ is binding, contradicting the uniqueness of $B(Y_{i,s}, Z')$ (Prop. 18).

Conversely, let $t(Z'_{i''}) \not\subseteq T'$. Define $Z'' = Z' \cup Y/Z'_{i''}$. As T' is full for Z', $|i_\subseteq(Z', T')| = \sum_{t\in T'} \bar{q}^s_t$. $i_\subseteq(Z'', T') = i(Z'', T) = i_\subseteq(Z', T') \cup \{i\}$ as $t(Z'_{i''}) \not\subseteq T'$, $i(Y) \notin i(Z')$, and $i(Y) \in T'$. So, $|i_\subseteq(Z'', T')| = |i_\subseteq(Z', T')| + 1 > \sum_{t\in T'} \bar{q}^s_t$. So, $Z' \cup Y/Z'_{i''}$ is not feasible.

Proof (Prop. 3). Assume Z' is feasible. When $i^* = \emptyset$, $Z'' = Z' \cup Y_{i,s}$ and is feasible. As Z'' is feasible, by Prop. 2 there is an $X'' \subseteq Z''$ with $i(X'') = i(Z'')$. We find an X''' such that $X''' \succ X'$ when $i(X') \subset i(Z'')$ and when $i(X') \not\subseteq i(Z'')$.
($i(X') \subset i(Z'')$) Take an $X' \subseteq Z'$. We construct $X''' \subseteq Z''$ with $|X'''_{T^p}| \geq |X'_{T^p}|$. Take $X^0 = X''$ and define X^l iteratively. If $|X^l_{T^p}| < |X'_{T^p}|$, there is a $t \in T^p$ such that $|X^l_t| < |X'_t|$ and an $i \in i(X'_t/X^l_t)$. Define $X^{l+1} =$

$X^l \cup X''_i/X^l_i$. Then each X^l is an allocation as $|X^{l+1}_t| \leq |X'_t + 1| \leq \bar{q}^s_t$ and $|X^{l+1}_{t'}| \leq |X^l_{t'}| \leq \bar{q}^s_{t'}$ for all $t' \in T/t$. When $|X^l_{T^p}| \geq |X'_{T^p}|$ define $X''' = X^l$. Each $i \in i(X'_{T^p}/X''_{T^p})$ is chosen at most once and the process concludes with $|X^l_{T^p}| \geq |X'_{T^p}|$.

For any $X' \in Z'$ and $X''' \in Z''$ with $|X'_{T^p}| = |X''_{T^p}|$, take the sequences $I' = \{i_1, \ldots, i_{|X'|}\}$ and $I'' = \{i_1, \ldots, i_{|X'|}, i'_1, \ldots, i'_{|X''|-|X'|}\}$ where $i_l \in i(X''')$ and $i'_l \in i(X''')/i(X')$. Then $X''' \succ_s X'$ as $i'_l \, \pi^s \, \emptyset$ for all $i'_l \in i(X''')$.
($i(X') \not\subseteq i(Z'')$). Here, $i^* \in i(X')$ so there is an $i'' \in i(B(Y_{i*}, Z''))/i(X')$. Let $Z''' = Z' \cup Y_{i,s}/Z_{i''}$.By $(i(X') \subset i(Z''))$, there is an X^4 such that $X^4 \subseteq Z'''$, $i(X^4) = i(Z''')$, and $X^4 \succsim_s X'$. Let $Y' = B(Y_{i''}, Z''')$, then $Y'' = Y' \cup \{i''\}/i^*(Y_{i''}, Z''')$ is feasible, so there is an allocation $X^5 \subseteq Y''$ with $i(X^5) = i(Y'')$. Define $X''' = X^5 \cup X^4_{T/t(B(Y_{i''},Z'''))}$. Here $|X'''_{T^p}| = |X^4_{T^p}|$. Take the sequences $I' = \{i'', i_2, \ldots, i_{|X'|}\}$ and $I'' = \{i^*, i_2, \ldots, i_{|X'|},\}$ where $i_l \in i(X''')$. Then $X''' \succ_s X^4$ as $i'' \, \pi^s \, i^*$, so $X''' \succ_s X'$

All X''' in $(i(X') \subset i(Z''))$ and $(i(X') \not\subseteq i(Z''))$ have same students, so the X''' that maximizes X'''_{T^p} is weakly preferred to all other X'''; therefore, it is strictly preferred to all X'.

Proof (Prop. 4). **($\Rightarrow$)** Let $X' \in \mathcal{C}^s(Y)$. Take $Z' = \{x \in Y_s : i(x) \in i(X')\}$. **(i)** By Prop. 1, Z' is feasible. **(ii)** If an $i' \in i(X')$ satisfies $\emptyset \, \pi^s \, i'$, take $X'' = X'/X'_{i'}$. $i(X'') = \{i_1, \ldots, i_{|X''|}\}$ dominates $i(X') = \{i_1, \ldots, i_{|X''|}, i'\}$ as $\emptyset \, \pi^s \, i'_l$. Contradicting $X' \in \mathcal{C}^s(Y)$. **(iii)** If $i \succ_s i^*$ for some $i \in i(Y_s)/(Z')$, then by Prop. 3 there is an $X'' \subseteq Z' \cup Y_i/Z'_{i^*,s}$ such that $X'' \succ_s X'$. Contradicting $X' \in \mathcal{C}^s(Y)$.

($\Leftarrow$) Assume (i), (ii), and (iii) are satisfied for some Z'. Then, as $\succsim_s$ is a preorder, there is a maximal element X' under the preorder $\succsim_s$. The proof of ($\Rightarrow$) shows that Z' with $i(Z') = i(X')$ satisfy (i), (ii), and (iii). We show that any Z' that satisfy the conditions have the same students. Assume Z' and Z'' satisfy (i), (ii), and (iii). Assume that $i(Z') \neq i(Z'')$, then there is a maximally ranked student in $i \in (i(Z')/(Z'')) \cup (i(Z'')/(Z'))$. WLOG let $i \in i(Z')/i(Z'')$. Let $i^* = i^*(Z'_i, Z'')$. Either $B(Z'_i, Z'') = \infty$ and $i^* = \emptyset$ or there is an $i' \in i(B(Z'_i, Z''))$ such that $i' \notin i(Z')$. If $i^* = \emptyset$ then $i \succ_s i^*$; otherwise, $i \succ_s i' \succsim_s i^*$. Both cases contradict (iii). Therefore, $Z' = Z''$ and there is an $X' \in Z'$ with $i(X') = i(Z')$ and $X' \in \mathcal{C}^s(Y)$.

Proof (Prop. 5). We show that Z' satisfies (i), (ii), and (iii) of Prop. 4. **(i)** We use induction. $Z^0 = \emptyset$ is feasible. So if Z^{l-1} is feasible, then by Prop. 2, Z^l is feasible. **(ii)** We show $i \succ_s \emptyset$ for all $i \in i(Z')$. This is trivially true for $Z^0 = \emptyset$. Assume $i \succ_s \emptyset$ for all $i \in i(Z^{l-1})$. Then $i^* = i^*(Y_{i_l,s}, Z^{l-1}) \succsim \emptyset$ as $i^* = \emptyset$ or $i^* \in i(Z^{l-1})$. So, if $i \in i(Z^l)$ then either $i \in Z^{l-1}$ and $i \succ_s \emptyset$ or $i = i_j$ and $i \succ_s i^* \succsim \emptyset$. **(iii)** We show that $i_l^* = i^*(Y_{i_k,s}, Z^l)$ is increasing in l. By (ii), $i_l^* \succsim_s \emptyset$ for all l. If $i_l^* = \emptyset$ then $i_{l+1}^* \succsim_s i_l^*$, so assume $i_l^* \neq \emptyset$. Define $T_l = t(B(Y_{i_k,s}, Z^l))$ and $I_l = i(B(Y_{i_k,s}, Z^l))$. If $T_{l+1} \subseteq T_l$, then $I_{l+1} \subseteq I_l \cup i_{l+1}$. Therefore, for any $i \in I_{l+1}$ either $i = i_{l+1} \succ_s i^*(Y_{i',s}, Z^l) \succsim_s i_l^*$ or $i \succsim_s i_l^*$; so, $i_l^* \succsim_s i_l^*$.

If $T_{l+1} \nsubseteq T_l$, then $i^*(Y_{i_l,s}, Z^l) \in I_l$. Let $B_{k,l} = B(Y_{i_k,s}, Z^l) \cup B(Y_{i_l,s}, Z^l))$. Then, $t(B_{k,l})$ is the union of two full sets; therefore, it is full. Further, $B(Y_{i_l,s}, Z^l) \subseteq B_{k,l}$; therefore, i_l and $i^*(Y_{i_l,s}, Z^l)$ so $B(Y_{i_l,s}, Z^l)$ remains full. As $i^*(Y_{i_l,s}, Z^l) \in I_l$, $i^*(Y_{i_l,s}, Z^l) \succsim_s i_l^*$. Therefore, $i' \succsim_s i_l^*$ for $i' \in B_{k,l} \cup i_l$. As, $B(Y_{i_k,s}, Z^{l+1}) \subseteq B_{k,l}$ the result follows.

Proof (Prop. 6). **(i,⇒)** Assume $\tilde{Z}'$ is feasible for T', then $\tilde{Z}'' = \tilde{Z}'_{T^p}$ is feasible for T' under hard bounds so there is an $X' \subseteq \tilde{Z}'' = \tilde{Z}'_{T^p}$ satisfying $i(X') = \{i \in i(\tilde{Z}') : t(\tilde{Z}'_{i,T^p}) \subseteq T'\} = \{i \in i(\tilde{Z}') : t(\tilde{Z}'_i) \cap T^p \subseteq T'\}$. **(i,⇐)** Assume there is an allocation $i(X') = \{i \in i(\tilde{Z}') : t(\tilde{Z}'_{i,T^p}) \subseteq T'\}$, then $\tilde{Z}'' = \tilde{Z}'_{T^p}$ is feasible for T^p under hard bounds; therefore, $\tilde{Z}'$ is feasible for T^p. **(ii,⇒)** Let Z' be feasible. Then $\tilde{Z}'$ is feasible for T^p; therefore, by (i) there is an $X'' \in \tilde{Z}'$ such that $i(X'') = i(\tilde{Z}')$. As X'' is an allocation, $|X''_t| \leq \bar{q}_t^s$ for all $t \in T^p$. Let $X''' = \hat{Z}'_{t_0}$. As $t_0 \in \hat{Z}'_{i',s}$ for all $i' \in i(\hat{Z}')$, $i(X''') = i(\hat{Z}')$. Let $X' = X'' \cup X'''$. Then $X'' \subseteq \tilde{Z}' \subseteq Z'$ and $X''' \subseteq \hat{Z}' \subseteq Z'$; therefore, $X' \subseteq Z'$. Further, $i(X') = i(X'') \cup i(X''') = i(\tilde{Z}') \cup i(\hat{Z}') = i(Z')$. Therefore, $|X'| = |i(X')| = |i(Z')| \leq \bar{q}^s$. **(ii,⇐)** Assume $X' \subseteq Z'$ is an allocation such that $i(X') = i(Z')$ and $i(X_{T^p}) = i(\tilde{Z}')$. As X' is an allocation $|i(Z')| = |i(X')| = |X'| \leq \bar{q}^s$. Further, by (i, ⇒), $\tilde{Z}'$ is feasible for T^p. Therefore, Z' is feasible.

Proof (Prop. 7). **(i,⇒)** Assume $B(Y', \tilde{Z}') = \infty$, then there is no full T' containing $t(Y')$. As the union of full sets is full (Corr. 2), there is a $t \in t(Y')$ with $t \notin T'$ for any full $T' \subseteq T^p$. Take some $x \in Y'$ with $t(x) = t$, and let $i = i(x)$. Define $i_{\subseteq}(Y, T') = \{i' \in i(Y) : t(Y) \subseteq T'\}$ and $\tilde{Z}'' = \tilde{Z}' \cup Y_i$. Then for any T' such that $t \notin T'$, $i_{\subseteq}(Z'', T') = i_{\subseteq}(Z', T') \leq \sum_{t \in T'} \bar{q}_t^s$. If $t \in T'$, then $i_{\subseteq}(Z', T') < \sum_{t \in T'} \bar{q}_t^s$; therefore, $i_{\subseteq}(Z'', T') \leq i_{\subseteq}(Z', T') + 1 \leq \sum_{t \in T'} \bar{q}_t^s$. **(i,⇐)** Assume $Z'' = \tilde{Z}' \cup Y_i$ is feasible then for any T' with $t(Y') \subseteq T'$, $i_{\subseteq}(Z'', T') < i_{\subseteq}(Z', T') \leq \sum_{t \in T'} \bar{q}_t^s$; so, T' is not full. Hence, $B(Y', \tilde{Z}') = \infty$. **(ii)** Assume $B(Y', \tilde{Z}') \neq \infty$ and take $i' \in i(B(Y', \tilde{Z}'))$. Let $Z'' = Z'/Z'_{i'}$. Then, by the proof of (i), $B(Y', \tilde{Z}'') = \infty$. Therefore, by (i) there exists an

$i \in Y'_i$ such that $Z'' \cup Y'_i = Z' \cup Y'_i / Z'_{i'}$ is feasible.

Proof (Prop. 8). **(i)** Let $\tilde{Z}^0 = \tilde{Z}'$ and assume $I'' = i(\hat{Z}') = \{i_1, \ldots, i_{|I'|}\}$. For $l \in \{1, \ldots, |I'|\}$, define $\tilde{Z}^l = \tilde{Z}^{l-1} \cup \hat{Z}'_{i_l}$ if $B(\hat{Z}'_{i_l}, \tilde{Z}^{l-1}) = \infty$ and $\tilde{Z}^l = \tilde{Z}^{l-1}$ if $B(\hat{Z}'_{i_l}, \tilde{Z}^{l-1}) \neq \infty$. Let $\tilde{Z}'' = \tilde{Z}^{|I'|}$ and $\hat{Z}'' = \hat{Z}'/\tilde{Z}'$.

Then $\tilde{Z}^0$ is feasible for T^p. Assume $\tilde{Z}^{l-1}$ is feasible for T^p. Define $i_{\subseteq}(Y, T') = |\{i \in i(Y) : t(Y_i) \subseteq T'\}|$. If $B(\hat{Z}'_{i_l}, \tilde{Z}^{l-1}) = \infty$, for $T' \subseteq t(\hat{Z}'_{i_l})$, $i_{\subseteq}(\tilde{Z}^{l-1}, T') < \sum_{t \in T'} \bar{q}^s_t$, so $i_{\subseteq}(\tilde{Z}^l, T') = i_{\subseteq}(\tilde{Z}^{l-1}, T') + 1 \leq \sum_{t \in T'} \bar{q}^s_t$, so $\tilde{Z}^l$ is feasible for T^p. If $B(\hat{Z}'_{i_l}, \tilde{Z}^{l-1}) \neq \infty$ then $\tilde{Z}^l = \tilde{Z}^{l-1}$ is feasible for T^p. As Z' is feasible, $|i(Z'')| = |i(Z')| \leq \bar{q}^s$; therefore, Z'' is feasible.

We show that Z'' is valid. Take $i_l \in \hat{Z}''$, then $B(\hat{Z}'_{i_l}, \tilde{Z}^{l-1}) \neq \infty$. As $i_{\subseteq}(\tilde{Z}^l, T')$ increases in l, so $B(\hat{Z}'_{i_l}, \tilde{Z}') \neq \infty$ and Z'' is valid.

(ii) Let Z' and Z'' be valid with $Z' = Z''$. We show that $|\tilde{Z}'| = |\tilde{Z}''|$. Let $T' \subseteq T^p$ and $T'' \subseteq T^p$ be the the union of the full sets in $\tilde{Z}'$ and $\tilde{Z}''$, respectively. By Corr. 2, T' and T'' are full under Z' and Z'', respectively. If $T' \neq T''$, WLOG let $T''' = T'/T'' \neq \emptyset$. As T' is full under Z', $I' = i_{\subseteq}(\tilde{Z}', T')/i_{\subseteq}(\tilde{Z}', T'')$ satisfies $|I'| \geq \sum_{t \in T'''} \bar{q}^s_t$. As T' is not full under Z'', $I'' = i_{\subseteq}(\tilde{Z}'', T')/i_{\subseteq}(\tilde{Z}'', T'')$ satisfies $|I''| < \sum_{t \in T'''} \bar{q}^s_t$. Therefore, there is some $i \in I'/I''$. So, $B(\hat{Z}''_i, \tilde{Z}'') = \infty$, contradicting the validity of Z''. Therefore, $T' = T''$. Now, let I' and I'' be the sets of students whose contract are not in a full sets in Z' and Z'', respectively. If there is an $i \in I'/I''$, then $B(\hat{Z}''_i, \tilde{Z}'') = \infty$, contradicting the validity of Z''. Since, Z' and Z'' are both full under T' and have the same students in T^p/T', $|\tilde{Z}'| = |\tilde{Z}''|$.

Proof (Prop. 9). **(i)** If $B(Y_{i,s}, \tilde{Z}') = \infty$, let $\tilde{Z}'' = \tilde{Z}' \cup Y_{i,s}$ and $\hat{Z}'' = \hat{Z}'$. By Prop 2, $\tilde{Z}''$ is feasible for T^p. As $|i(Z'')| = |i(Z')| + 1 \leq \bar{q}^s$, Z'' is feasible. Z' is valid, so $B(\hat{Z}', \tilde{Z}')) \neq \infty$. $B(\hat{Z}', \tilde{Z}'') = B(\hat{Z}', \tilde{Z}'') \neq \infty$ as $t(Y_{i,s}) \nsubseteq B(\hat{Z}', \tilde{Z}')$; therefore, Z'' is valid. If $B(Y_{i,s}, \tilde{Z}') \neq \infty$ then there is an $i' \in i \cup i(B(Y_{i,s}, \tilde{Z}'))$ with $t_0 \in t(Y_{i',s})$. Take $\tilde{Z}'' = \tilde{Z}' \cup Y_{i,s}/Y_{i',s}$ and $\hat{Z}'' = \hat{Z}' \cup Y_{i',s}$. By Prop 2, $Z'' = Z'$ is feasible for T^p. Further, $|i(Z'')| = |i(Z')| + 1 \leq \bar{q}^s$, so Z'' is feasible. As Z' is valid, $t(B(\hat{Z}', \tilde{Z}')) = T'$ for some full T'. Further, $t(Y_{i',s}) \cup T^p \subseteq T''$ for some full T''; therefore, $(\hat{Z}' \cup Y_{i',s}) \cup T^p \subseteq T' \cup T''$ and Z'' is valid. **(ii)** As $Span(Y_{i,s}, Z') = \infty$ and $|i(Z')| = \bar{q}^s$, $B(Y_{i,s}, \tilde{Z}') = \infty$. If $i' \in i(\hat{Z}')$, let $\tilde{Z}'' = \tilde{Z}' \cup Y_{i,s}$ and $\hat{Z}'' = \hat{Z}'/\hat{Z}'_{i'}$. As $B(Y_{i,s}, \tilde{Z}') = \infty$, $\tilde{Z}''$ is feasible for T^p by Prop 2. As $|i(Z'')| = |i(Z')| = \bar{q}^s$, Z' is feasible. $t(B(\hat{Z}'', \tilde{Z}'')) \subseteq t(B(\hat{Z}', \tilde{Z}'))$ as $\hat{Z}'' \subseteq \hat{Z}'$ and $\tilde{Z}' \subseteq \tilde{Z}''$. Therefore, the validity of Z' implies Z'' is valid. If $i' \in i(\tilde{Z}')$ then $i' \in i(B(\hat{Z}', \tilde{Z}''))$. By Prop.

7, there is an $i'' \in i(\hat{Z}')$ such that $\tilde{Z}''' = \tilde{Z}' \cup \hat{Z}'_{i'',s}/\tilde{Z}'_{i'}$ is feasible for T^p. Further, $B(Y_{i,z}, Z''') = \infty$ so $Z'' = \tilde{Z}' \cup \hat{Z}'_{i'',s} \cup Y_{i,s}/\tilde{Z}'_{i's}$ is feasible for T^p. Let $\hat{Z}'' = \hat{Z}'/\hat{Z}'_{i'',s}$. Here, $|i(Z'')| = |i(Z')| = \bar{q}^s$. As Z' is valid, $B(\hat{Z}', \tilde{Z}') \neq \infty$. As $\hat{Z}'' \subseteq \hat{Z}'$ and $t(\hat{Z}'_{i'',s}) \in B(\hat{Z}', \tilde{Z}')$, $t(B(\hat{Z}'', \tilde{Z}'')) \subseteq t(B(\hat{Z}', \tilde{Z}')) \neq \infty$, so Z'' is valid.
($Z'' \succsim_s Z'''$, i) $Z''' \subset Z''$ as $Z''' \neq Z''$; therefore, by Lemma 1, $Z'' \succ_s Z'''$.
($Z'' \succsim_s Z'''$, ii) Take $i'' \in i(Z' \cup Y_{i,s})/i(Z''')$. If $i'' \in Span(Y_{i,s}, Z')$, then $|i(\tilde{Z}''')| = |i(\tilde{Z}'')|$ by the proof of (ii). As $i \, \pi^s \, i'$, $\{i'', i_2, \ldots, i_{|i(Z'')|}\} = i(Z')$ dominates $\{i', i_2, \ldots, i_{|i(Z'')|}\} = i(Z''')$. If $i'' \notin Span(Y_{i,s}, Z')$, then, defining $T' = t(Span(Y_{i,s}, Z'))$, $T' \cap T^p$ is full under Z'''. The feasibility of Z''' implies $t_0 \in T'$; otherwise, Z''' is not feasible. Define $i_{\subseteq}(Y, T') = \{i \in i(Y) : t(Y_i) \subseteq T' \cup \{t_0\}\}$. As Z'' and Z''' are valid, $|i_{\subseteq}(\tilde{Z}'', T')| = \sum_{t \in T'} \bar{q}^s = |i_{\subseteq}(\tilde{Z}''', T')|$. As $i'' \notin Span(Y_{i,s}, Z')$, $i(\tilde{Z}''')/i_{\subseteq}(\tilde{Z}''', T') = i(\tilde{Z}'')/(i_{\subseteq}(\tilde{Z}'', T') \cup i'')$; therefore, $|i(\tilde{Z}'')| > |i(\tilde{Z}''')|$ and $Z'' \succ_s Z'''$. For any Z''' such that $|i(Z''')| < \bar{q}^s$, there is an Z^4 with $|i(Z^4)| = \bar{q}^s$. By Lemma 1, $Z'' \succsim_s Z^4 \succ_s Z'''$.

Proof (Prop. 10). **(i)** We show Z'' is feasible when (a) $i^* \in i(B(Y_{i,s}), \tilde{Z}')$ (b) $i^* \in i(\tilde{Z}')$, and (c) $i^* \in i(B(\hat{Z}', \tilde{Z}')/B(Y_{i,s}, \tilde{Z}'))$. **(a)** If $i^* \in i(B(Y_{i,s}, \tilde{Z}'))$, $\tilde{Z}'' = \tilde{Z}' \cup Y_{i,s}/Y_{i^*,s}$ is feasible for T^p by Prop. 2. As $|i(\tilde{Z}'')| = |i(\tilde{Z}'')| \leq \bar{q}^s$, Z'' is feasible. **(b)** If $i^* \in i(\tilde{Z}')$, there is an $i' \in i \cup i(B(Y_{i.s}, \tilde{Z}'))$ with $t_0 \in t(Y_{i',s})$. Let $\tilde{Z}'' = \tilde{Z}' \cup Y_{i,s}/Y_{i',s}$ and $\hat{Z}'' = \hat{Z}' \cup Y_{i',s}/Y_{i^*,s}$. By Prop. 2, $\tilde{Z}''$ is feasible for T^p. As $|i(Z'')| = |i(Z')| = \bar{q}^s$, Z'' is feasible. **(c)** When $i^* \in i(B(\hat{Z}', \tilde{Z}')/B(Y_{i,s}, \tilde{Z}'))$, then there exists an $i'' \in \hat{Z}'$ such that $\tilde{Z}' \cup \hat{Z}'_{i''}/\tilde{Z}'_{i^*,s}$ is feasible, by Prop. 7. If $i^* \in i(\tilde{Z}')$, there is an $i' \in i \cup i(B(Y_{i.s}, \tilde{Z}'))$ with $t_0 \in t(Y_{i',s})$. Let $\tilde{Z}'' = \tilde{Z}' \cup Y_{i,s} \cup \hat{Z}'_{i''}/Y_{i',s}/\tilde{Z}'_{i^*,s}$ and $\hat{Z}'' = \hat{Z}' \cup Y_{i',s}/Y_{i''}$. As $\tilde{Z}'' = \tilde{Z}''' \cup Y_{i,a}/\tilde{Z}'_{i',s}$ where $\tilde{Z}''' = \tilde{Z}' \cup \hat{Z}'_{i''}/\tilde{Z}'_{i^*,s}$, $\tilde{Z}''$ is feasible for T^p by Prop. 2. As $|i(Z'')| = |i(Z')| \leq \bar{q}^s$, Z'' is feasible. We show Z'' is valid. Define $i_{\subseteq}(Y, T') = \{i' \in i(Y) : t(Y) \subseteq T' \cup \{t_0\}\}$ and let $T' = t(Span(Y_{i,s}, Z')) \cap T^p$. Here, $|i_{\subseteq}(Z', T')| = |i_{\subseteq}(Z'', T')|$ so Z'' is full under T'. As $t(\hat{Z}'') \cap T^p \subseteq T'$, $B(\hat{Z}'', \tilde{Z}'') \neq \infty$.

($Z'' \succsim_s Z'''$) $|\tilde{Z}''| \geq |\tilde{Z}'''|$ for all valid $Z''' \in Z' \cup Y_{i,s}$; otherwise, there is an $i \in i(\tilde{Z}''')/i(\tilde{Z}'')$ with $B(\tilde{Z}'''_i, \tilde{Z}'') = \infty$, contradicting the validity of Z''. Let $i' \in i(Z') \cup i/i^*$. We show Z'' is preferred to any $Z''' = Z' \cup Y_{i,s}/Z'_{i'}$. When $i' \in i(Span(Y_{i,s}, Z'))$, Z''' is a substitution of the form (a), (b), (c). Therefore, when Z''' is valid, $|i(\tilde{Z}''')| = |i(\tilde{Z}')| = |i(\tilde{Z}'')|$. By applying the remainder of the proof of ($Z'' \succsim_s Z'''$, ii) from Prop. 9, it follows that $Z'' \succ_s Z'''$. For any Z''' such that $|i(Z''')| < \bar{q}^s$, there is an Z^4 with $|i(Z^4)| = \bar{q}^s$. By Lemma 1,

$Z'' \succsim_s Z^4 \succ_s Z'''$.

Proof (Prop. 11). **(⇒)** For $X' \in \mathcal{C}^s(Y)$, define $Z' = \{x \in Y_s : i(x) \in i(X')\}$, $\hat{Z}' = \{x \in Y_s : i(x) \in i(X'_{t_0})\}$, and $\tilde{Z}' = Z'/\hat{Z}'$. **(i)** As X'/X'_{t_0} is an allocation, $\tilde{Z}'$ is feasible for T^p. As $|i(Z')| = |i(X')| \leq \bar{q}^s$, Z' is feasible. Z' is valid; otherwise, there is a Z'' with $|i(\tilde{Z}'')| > |i(\tilde{Z}')|$. By Prop. 6 there is an $X'' \subseteq Z''$ with $|X''/X''_{t_0}| = |i(\tilde{Z}'')| > |i(\tilde{Z}')| = |X'/X'_{t_0}|$. **(ii)** If there is an $i' \in i(X')$ with $\emptyset \, \pi^s \, i'$, then $X'' = X'/X'_{i'} \succ_s X'$ as $i(X'') = \{i_1, \ldots, i_{|X''|}\}$ dominates $i(X') = \{i_1, \ldots, i_{|X''|}, i'\}$, contradicting $X' \in \mathcal{C}^s(Y)$. **(iii)** Let $i \in i(Y_s/Z')$ such that $i \, \pi^s \, i^*(Y_{i,s}, Z')$; then either $i^*(Y_{i,s}, Z') = \emptyset$ or $i^*(Y_{i,s}, Z') \neq \emptyset$. If $i^*(Y_{i,s}, Z') = \emptyset$, Prop. 9 implies there exists an $Z'' \succ_s Z'$. If $i^*(Y_{i,s}, Z') \neq \emptyset$, Prop. 10 implies there exists an $Z'' \succ_s Z'$. In either case there is an $X'' \subseteq Z''$ satisfying $X'' \succ_s X'$, contradicting $X' \in \mathcal{C}^s(Y)$.

(⇐) Assume (i), (ii), and (iii) are satisfied for some Z'. Then, as $\succsim_s$ is a preorder, there is a maximal element X' under the preorder $\succsim_s$. The proof of (⇒) shows that Z' with $i(Z') = i(X')$ satisfy (i), (ii), and (iii). We follow the procedure in the proof of Prop. 4. Let Z' and Z'' satisfy (i), (ii), and (iii). Assume that then there is a maximally ranked student in $i \in (i(Z')/(Z'')) \cup (i(Z'')/(Z'))$. WLOG let $i \in i(Z')/i(Z'')$. Then as Z' and Z'' satisfy (iii), $i^* = i^*(Z'_i, Z'')$. We assume $i^* \, \pi^s \, i$. Then $Span(Y_{i,s}, \tilde{Z}'') \subseteq Z'$. However, $Span(Y_{i,s}, \tilde{Z}'') \neq \infty$; therefore, either $Span(Y_{i,s}, \tilde{Z}'') = \infty$ or $Span(Y_{i,s}, \tilde{Z}'') \nsubseteq Z'$, a contradiction of $i^* \, \pi^s \, i$. Therefore, $(i(Z')/(Z'')) \cup (i(Z'')/(Z')) = \emptyset$ and $Z' = Z''$ and there exists an $X' \in Z'$ with $i(X') = i(Z')$ and $X' \in \mathcal{C}^s(Y)$.

Proof (Prop. 12). We show that Z' satisfies (i), (ii), and (iii) of Prop. 11. **(i)** We use induction. $Z^0 = \emptyset$ is feasible. So if Z^{l-1} is feasible, then by Props. 9 and 10, Z^l is feasible. **(ii)** We show $i \succ_s \emptyset$ for all $i \in i(Z')$. This is trivially true for $Z^0 = \emptyset$. Assume $i \succ_s \emptyset$ for all $i \in i(Z^{l-1})$. Then $i^* = i^*(Y_{i_l,s}, Z^{l-1}) \succsim \emptyset$ as $i^* = \emptyset$ or $i^* \in i(Z^{l-1})$. So, if $i \in i(Z^l)$ then either $i \in Z^{l-1}$ and $i \succ_s \emptyset$ or $i = i_j$ and $i \succ_s i^* \succsim \emptyset$. **(iii)** We show that $i^*_l = i^*(Y_{i_k,s}, Z^l)$ is increasing in l. By (ii), $i^*_l \succsim_s \emptyset$ for all l. If $i^*_l = \emptyset$ then $i^*_{l+1} \succsim_s i^*_l$, so assume $i^*_l \neq \emptyset$. Define $T_l = t(Span(Y_{i_k,s}, Z^l))$ and $I_l = i(Span(Y_{i_k,s}, Z^l))$. If $T_{l+1} \subseteq T_l$, then $I_{l+1} \subseteq I_l \cup i_{l+1}$. Therefore, for any $i \in I_{l+1}$ either $i = i_{l+1} \succ_s i^*(Y_{i',s}, Z^l) \succsim_s i^*_l$ or $i \succsim_s i^*_l$; so, $i^*_l \succsim_s i^*_l$.

If $T_{l+1} \nsubseteq T_l$, then $i^*(Y_{i_l,s}, Z^l) \in I_l$ so $i^*(Y_{i_l,s}, Z^l) \succsim_s i^*(Y_{i_k,s}, Z^l)$. Let $S_{k,l} = Span(Y_{i_k,s}, Z^l) \cup Span(Y_{i_l,s}, Z^l))$. By P rop. 5 $B(Y_{i_k,s}, \tilde{Z}^{l+1}) \subseteq B(Y_{i_k,s}, \tilde{Z}^l) \cup B(Y_{i_l,s}, \tilde{Z}^l)$ and $B(\hat{Z}, \tilde{Z}^{l+1}) \subseteq B(\hat{Z}^l, \tilde{Z}^l)$ if $t_0 \in S_{k,l}$; therefore,

$Span(Y_{i_k,s}, Z^{l+1}) \subseteq S_{k,l}$. As $i^*(Y_{i_l,s}, Z^l) \in I_l$, $i^*(Y_{i_l,s}, Z^l) \succsim_s i_l^*$. Therefore, $i' \succsim_s i_l^*$ for $i' \in S_{k,l} \cup i_l$. As, $Span(Y_{i_k,s}, Z^{l+1}) \subseteq S_{k,l}$ the result follows.

Proof (Prop. 13). **(Substitutability, ULS)** Example 7 show that $\mathcal{C}^s$ violates substitutability and ULS. **(BLS)** This is a direct result of Prop. 5 for hard bounds and Prop. 12 for soft bounds.

Proof (Prop. 14). Let $Y \subset X$, $x \in X/Y$, and $x \notin X'$ for all $X' \in \mathcal{C}^s(Y \cup \{x\})$. If $x \notin Y_s$, then $\mathcal{C}^s(Y) = \mathcal{C}^s(Y \cup \{x\})$ by definition, so let $x \in Y_s$.

Let $X'' \in \mathcal{C}^s(Y)$, then $X'' \succsim_s X'''$ for all $X''' \in Y_s$. As $x \notin X'$ for any $X' \in \mathcal{C}^s$, there exist an X^4 with $x \notin X^4$ such that $X^4 \in Y_s \cup \{x\}$. As $x \notin X^4$, $X^4 \in Y_s$. Since $X'' \in \mathcal{C}^s(Y)$ and $X'' \succsim_s X^4$, $X'' \in \mathcal{C}^s(Y \cup \{x\})$. Alternatively, let $X'' \in \mathcal{C}^s(Y \cup \{x\})$, $X'' \succsim_s X'''$ for all $X''' \in Y \cup \{x\}$; therefore, $X'' \succsim_s X'''$ for all $X''' \in$. Since $x \notin X''$, $X'' \in Y$ and $X'' \in \mathcal{C}^s(Y)$. Therefore, $\mathcal{C}^s(Y) = \mathcal{C}^s(Y \cup \{x\})$.

Proof (Prop. 15). For $(X, \{\succsim_i\}_{i \in I}, \{\succsim_s\}_{s \in S})$ let $\bar{t}(x) = \{t(x) : x' \sim_i x\}$ and $\bar{X} = \{(i, s, \bar{T}) : x \in X\}$. Say $Y \in \bar{Y}$, if for each $\bar{x} \in \bar{Y}$, there is an $x \in Y$ such that $i(x) = i(\bar{x})$, $s(x) = s(\bar{x})$, and $t(x) \in t(\bar{x})$. Define $\bar{C}^i(\bar{Y}) = \{\bar{x} \in \bar{Y} : x \in \mathcal{C}^i(Y), \text{ for some } x \in \bar{x}\}$, $\bar{\mathcal{C}}^s(\bar{Y}) = \{\bar{X}' \subseteq \bar{Y} : X' \in \mathcal{C}^i(Y), \text{ for } X' \in \bar{X}'\}$, and $\bar{C}^s(\bar{Y}) = \bar{C}^i(\bar{\mathcal{C}}^s(\bar{Y}))$. Then $\bar{C}^s$ satisfies IRC otherwise there is a $Y \in \bar{Y}$ and an $x \in \bar{x}$ such that $x \notin X'$ for some $X' \in \mathcal{C}(Y \cup Y' \cup \{x\})$ where $\mathcal{C}^s(Y \cup Y') \neq \mathcal{C}^s(Y \cup Y' \cup \{x\})$ for $Y' \in \{x' \in \bar{x}\}/x$, a contradiction. Similarly, $\bar{C}^s$ satisfies BLS, otherwise, by using IRC to repetitively remove unchosen $x'' \in \bar{x}'$, there is a $Y \in \bar{Y}$, $x \in \bar{x}$, and a $x' \in \bar{x}'$ such that $x \notin X'$ for some $X' \in \mathcal{C}^s(Y \cup \{x\})$ and $x \in X'$ for some $X' \in \mathcal{C}^s(Y \cup \{x, x'\})$, a contradiction. Using the result from Hatfield and Kojima (2010) and Aygün et al. (2012) there is a stable allocation in $\bar{X}' \in \bar{X}$. Therefore, $\bar{C}^i(\bar{X}') = \bar{X}'$ so $x \in \mathcal{C}^i(x)$ for all $x \in \bar{X}'_i$ and $X_s \in \mathcal{C}^s(X')$ for some $X' \in \bar{X}'$. If $X'' \in \mathcal{C}^s(X' \cup X'')$ and $X''_i \in \mathcal{C}^i(X' \cup X'')$ then $\bar{X}'' \in \bar{C}^s(\bar{X}' \cup \bar{X}'')$ and $\bar{X}''_i \in \mathcal{C}^i(\bar{X}' \cup \bar{X}'')$, a contradiction.

Proof (Prop. 16). **(i, Stability)** Assume X' is an adaptive acceptance algorithm allocation (AAA). As $X'_i \in Y_i^k$ for some k, $X_i \succsim_i \emptyset$; therefore, $X'_i \in \mathcal{C}^i(X')$. As $X'_s \in \mathcal{C}^s(Y)$ for the terminal Y in the AAA, $X'_s \succsim_s X''_s$ for any $X'' \subseteq Y$; therefore, $X'_s \succsim_s X''_s$ for any $X'' \subset X'$. Therefore, $X'_s \in \mathcal{C}(X')$. Therefore, X' is IR. Let $X'' \in C^s(X' \cup X'')$ such that $X'' \succ_s X'$, then $X'' \notin Y$; otherwise, $X'' \sim_s X'$. Therefore, there exists an $i \in i(X'')$ such that X''_i has not been considered by the AAA; therefore, $X'_i \succ X''_i$. Therefore, X' is UB. **(Student optimal)** Let $X'' \succ_i X'$ for some $i \in I$. Then there is some stage in the AAA where X''_i is rejected. WLOG, take i to be a student that is rejected at the earliest stage that a contract in X'' is rejected. Then letting Y' be the contracts at the start of the earliest rejection stage $i \notin \mathcal{C}^s(Y')$ so $i \notin \mathcal{C}^s(Y' \cup X')$ as $Y \subset Y' \cup X'$.

Lemma 1. *Let Z' be a valid partition such that $i \,\pi^s\, \emptyset$ for all $i \in i(Z')$. Then $Z' \succ_s Z''$ for any valid $Z'' \subset Z'$.*

Proof (Lemma 1). Let $\Delta Z_s = \tilde{Z}''/\tilde{Z}''$. Then $\Delta Z_s \subseteq \hat{Z}'$, so $B(\Delta Z_s, \tilde{Z}') \neq \infty$. Take $T' = t(B(\Delta Z_s, \tilde{Z}'))$. Define $i_{\subseteq}(Y, T') = \{i' \in i(Y) : t(Y) \subseteq T' \cup \{t_0\}\}$. Then as Z' is valid, $|i_{\subseteq}(\tilde{Z}', T')| = \sum_{t \in T'} \bar{q}^s \geq |i_{\subseteq}(\tilde{Z}'', T')|$. Further, as Z' is valid, if $i \in i(\tilde{Z}'')/i_{\subseteq}(\tilde{Z}'', T')$ then $i \in i(\tilde{Z}')/i_{\subseteq}(\tilde{Z}', T')$. Therefore, $|i(\tilde{Z}')| \geq |i(\tilde{Z}'')|$. If $|i(\tilde{Z}')| > |i(\tilde{Z}'')|$, then $Z' \succ_s Z''$. If $|i(\tilde{Z}')| = |i(\tilde{Z}'')|$, take a sequence $\{i_1, ..., i_{|i(Z'')|}\} = i(Z'')$. This sequence is dominated by the sequence $\{i_1, \ldots, i_{|i(Z'')|}, i'_1, \ldots, i'_{|i(Z')/i(Z'')|}\} = i(Z')$; therefore, $Z' \succ_s Z''$.

Appendix B. Additional Examples

Example 8 shows that students with a preference to being allocated a general status may lead to over-representation of the privileged group.

Example 8. Let $S = \{s\}$, $I = \{i_1, i_2, i_3, i_4\}$, and $T = \{t_0, t_1\}$. Statuses are $\tau(i_1) = \tau(i_3) = T$ and $\tau(i_2) = \tau(i_4) = \{t_0\}$. The quotas are $\bar{q} = 2$ with $\bar{q}_{t_1} = 1$. Let $i_1 \,\pi\, i_2 \,\pi\, i_3 \,\pi\, i_4$. When $(i_1, t_0) \succ_i (i_1, t_1)$, $X' = \{(i_1, t_0), (i_3, t_1)\}$ and $X'' = \{(i_1, t_1), (i_2, t_0)\}$ are stable. Under the student optimal differed acceptance algorithm, X' is chosen, leading to over-representation.

Appendix C. Technical Appendix

Appendix C.1. $\succsim_s$ is a preorder

Proposition 17. *$\succsim_s$ is preorder.*

Proof (Prop. 17). We show that $\succsim_s$ is reflexive and transitive.
Reflexive: Assume $X' \subseteq X$ then $q^s_p(X') = q^s_p(X'')$ and the sequence $x_1, \ldots, x_{|X'|} \in X'$ and $x_1, \ldots, x_{|X'|} \in X'$ have $x_l = x_l$ for all $l \in \{1, \ldots, |X'|\}$. *Transitive:* Assume $X' \succsim_s X''$ and $X'' \succsim_s X'''$, then $q^s_p(X') \geq q^s_p(X'') \geq q^s_p(X''')$. If

$q^s_p(X') > q^s_p(X''')|$ then $X' \succsim_s X'''$. Otherwise, $q^s_p(X') = q^s_p(X'') = q^s_p(X''')$ and $|X'| \geq |X''| \geq |X'''|$ and there are non-repeating sequences such that $x_1, \ldots, x_{|X''|} \in X'$ and $x'_1, \ldots, x'_{|X''|} \in X''$ such that $x_l \, \pi^s \, x'_l$ or $x_l = x'_l$ for all $l \in \{1, \ldots, |X''|\}$, and $y''_1, \ldots, y''_{|X''|} \in X'$ and $y''''_1, \ldots, y'''_{|X'''|} \in X'''$ such that $y''_m \, \pi^s \, y'''_m$ or $y''_m = y'''_m$ for all $m \in \{1, \ldots, |X'''|\}$. Fix the index m and reorder the index l such that $x'_l = y''_m$ for all $l \leq |X'''|$. As $x'_1, \ldots, x'_{|X''|}$ contains all $x' \in X''$, this is possible. Then $x_1, \ldots, x_{|X'''|}$ and $y''''_1, \ldots, y'''_{|X'''|}$ are sequences such that either $x_l \, \pi^s \, y'''_l$ or $x_l = y'''_l$, as $x_l = x'_l = y'''_l$, $x_l \, \pi^s \, x'_l = y'''_l$, $x_l = x'_l \, \pi^s \, y'''_l$, or $x_l \, \pi^s \, x'_l \, \pi^s \, y'''_l$.

Proposition 18. *Assume Z' is feasible for T^p. Let $Y'_s \subseteq X'_s$ with $i(Y'_s) \cap i(Z) = \emptyset$, then $B(Y'_s, Z')$ is well-defined.*

Proof (Prop. 18). If there is no T' such that Z' is full for T' and $t(Y_s) \subseteq T'$, then $B(Y'_s, Z') = \emptyset$. So, let T' and T'' be full sets of Z' that contain $t(Y'_s)$. We show that $T_1 \cap T_2$ is also full and contains $t(Y'_s)$, so $T_1 = T_2$. Define $\tilde{T}' = T'/(T' \cap T'')$, $q^s_{T' \cap T''} = |\{i \in i(Z') : t(Z'_i) \subseteq T'''\}|$, and $q^s_{T'''} = |\{i \in i(Z') : t(Z'_i) \subseteq T''', t(Z'_i) \nsubseteq T' \cap T''\}|$ for any $T''' \subseteq \{T', T''\}$. As Z' is feasible,

$$q^s_{\tilde{T}'} + q^s_{T' \cap T''} + q^s_{\tilde{T}''} \leq \sum_{t \in \tilde{T}'} \bar{q}^s_t + \sum_{t \in T' \cap T''} \bar{q}^s_t + \sum_{t \in \tilde{T}''} \bar{q}^s_t$$

Subtracting $q^s_{T'''} + q^s_{T' \cap T''} = \sum_{t \in T'''} \bar{q}^s_t + \sum_{t \in T' \cap T''} q^s_t$ for $T''' \in \{\tilde{T}', \tilde{T}''\}$ from both sides gives $-q^s_{T' \cap T''} \leq -\sum_{t \in T' \cap T''} \bar{q}^s_t$ or $q_{T' \cap T''} \geq \sum_{t \in T' \cap T''} \bar{q}^s_t$. Further, $t(Y'_s) \subseteq T'$ and $t(Y'_s) \subseteq T''$, $t(Y'_s) \subseteq T' \cap T''$, so the binding set is unique, and hence well-defined.

Corollary 2. *Assume Z' is full for T' and T''. Then $T' \cap T''$ and $T' \cup T''$ are full for Z'.*

Proof (Corr. 2). Proposition 18 shows that $T''' = T' \cap T''$ is full for Z'. Using the notation from Proposition 18, we have

$$q^s_{\tilde{T}'} + q^s_{T' \cap T''} = \sum_{t \in \tilde{T}'} \bar{q}^s_t + \sum_{t \in T' \cap T''} \bar{q}^s_t$$

as T', T'' and T''' are full. Therefore, $q^s_{\tilde{T}'} = \sum_{t \in \tilde{T}'} \bar{q}^s_t + \sum_{t \in T' \cap T''} \bar{q}^s_t - q^s_{T' \cap T''}$. Similarly, $q^s_{\tilde{T}''} = \sum_{t \in \tilde{T}''} \bar{q}^s_t + \sum_{t \in T' \cap T''} \bar{q}^s_t - q^s_{T' \cap T''}$

Defining $q^s_{T'\cup T''} = |\{i \in i(Z') : t(Z'_i) \subseteq T' \cup T''\}|$

$$q^s_{T'\cup T''} \geq q^s_{\tilde{T}'} + q^s_{T'\cap T''} + q^s_{\tilde{T}''} = \sum_{t\in\tilde{T}'} \bar{q}^s_t + \sum_{t\in T'\cap T''} \bar{q}^s_t + \sum_{t\in\tilde{T}''} \bar{q}^s_t$$

So $T' \cup T''$ is full.