Proportional Rules for State Contingent Claims*

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Abstract

We consider rationing problems where the claims are state contingent. Before the state is realized individuals submit claims for every possible state of the world. A rule distributes resources before the realization of the state of the world. We introduce two natural extensions of the proportional rule in this framework, namely, the ex-ante proportional rule and the ex-post proportional rule, and then we characterize them using standard axioms from the literature.

Keywords: Rationing, Proportional rule, State contingent claims, No advantageous reallocation

JEL Classification: C71, D63, D81

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1 Introduction

The rationing problem is arguably the simplest model of distributive justice. The problem involves a resource that is to be divided among individuals each of whom submit a claim for the resource. Rationing is required when the sum of the claims exceeds the resource, with typical examples being bankruptcy, taxation, inheritance, etc. The problem of rationing is as old as the history of civilization itself. We can find examples of such problems in ancient texts such as the Talmud and Aristotle. The first formal analysis of the rationing problem was presented by O'Neill (1982) where he descibes the resource as an inheritance. The problem of rationing is an ethical or normative issue since neither the market nor traditional institutions can provide convincing solutions. For this reason, adopting an axiomatic approach has been the focus of the literature on rationing.

The most natural rule in this context probably arises from Aristotle's maxim, "Equals should be treated equally, and unequals unequally, in proportion to relevant similarities and differences" from Nicomachean Ethics. The *proportional* rule gives shares in proportion to claims. There are various normative treatments of the Proportional rule, such as O'Neill (1982), Moulin (1987), Chun (1988), Young (1988), and Ju et al. (2007), etc.

Other rules central to the literature are based on normative axioms, including various forms of egalitarianism. The *uniform gains* rule equalizes the shares such as the shares do not exceed the claims. The *uniform losses* rule equalizes the losses (difference between claim and share) to the extent that it is possible. One can refer to some axiomatic characterizations of egalitarian rules for different environments in Dagan (1996), Herrero and Villar (2001), Sprumont (1991), Kesten (2006), Juarez and Kumar (2013), etc. Young (1987a) characterizes a class of parametric rules in the taxation problem and Young (1987b) introduces another important family of rules called the *equal sacrifice* rules. Rules from ancient texts and their extensions have also been considered by various authors. Aumann and Maschler (1985) provides a rule from the Talmud in the bankruptcy context, and papers like Hokari and Thomson (2003) study generalizations of the same. Alcalde, Marco, and Silva (2005) extends an old solution for bankruptcy problems described by Ibn Ezra in the 12^{th} century. Surveys of rationing problems are provided by Moulin (2002) and Thomson (2003, 2013) where interesting characterizations of rationing rules are mentioned.

We consider the rationing problem in a two-stage setting where the claims are state contingent. In the first stage, each individual submits a claim for every possible state of the world. The realization of the state happens in the second stage. A rule must distribute the resources in the first stage, i.e., before the realization of the state of the world. Such a situation may arise, for instance, in the allocation of the fiscal budget of a country. Different departments of a government may require different resources in different states of the world to be realized in the coming fiscal year. For example, the Department of Defense may have different requirements depending on its relations with neighboring countries in the following year. The Department of Agriculture has requirements based on factors like rainfall next year. The Department of Health may have requirements that depend on factors like the incidence of epidemics and the weather. However, the federal budget must be allocated at the beginning of the fiscal year.

Another example of our setting is the distribution of research funds (or travel grants) among graduate students of a department in a university who expect travel or research expenses contingent on the state of the world (e.g., expenses based on the results of their research, travels based on the conferences accepting their paper, etc.). A situation like our setting also arises in the allocation of university funds among different departments based on their performance, or need, or NSF funds to researchers from various universities.

This natural framework of two-stage, state-contingent rationing problem has not been given much consideration in the literature. A fairly close setting called multi-issue allocations (MIA), introduced in Calleja et al. (2005), has been studied. In MIA, the claim of each individual is a vector that specifies the amount claimed on each issue and a rule distributes shares for each issue and each individual. Bergantiños et al. (2011), and Lorenzo-Freire et al. (2010) provide several axiomatic characterizations of uniform gains and uniform losses rules in MIA whereas Moreno-Ternero (2009), and Bergantiños et al. (2010) provide axiomatic characterizations of the proportional rule in MIA. The MIA framework, however, does not consider uncertainty.

A similar framework to ours that considers uncertainty has been analyzed by Habis and Herings (2013). They focus on the stability[1] of the stochastic ex-

[1] They used "Weak Sequential Core" as the stability criterion which was defined in Habis and Herings (2011).

tensions of various rationing rules and show that the only stable rule is the stochastic extension of the uniform gains rule. Xue (2015) studies the egalitarian rule for the pre-committed division of a perfectly divisible commodity where the claims are uncertain. Chun and Thomson (1990a and 1990b) study the bargaining problem with uncertain disagreement points. Although the bargaining problem and the rationing framework are similar, they differ in the following sense. Bargaining theory deals with feasible sets that are arbitrarily convex and compact sets as opposed to being the comprehensive hull of subsets of a hyperplane normal to a vector of ones.

In our two-stage framework, the individuals submit their claims in the first stage and realization of the state happens in the second stage. The resource must be allocated in the first stage. Two particularly natural approaches arise in such situations. The first approach is to apply a rationing rule on the expectation of the claims, which we call the *ex-ante* rationing rule. In the second approach, we first use a rationing rule to find the shares of every individual for each state based on the claim profile in that state. Then the final shares are calculated by taking the expectation of the shares over states. We call this rule the *ex-post* rationing rule.

In this paper, we will focus our attention on proportional rules and characterize both the ex-ante proportional rule and the ex-post proportional rule. Our axiomatic characterizations are based on the *No Advantageous Reallocation* axiom introduced by Moulin (1985). This axiom states that no group of individuals can benefit from reallocating their claims amongst themselves. We extend this concept to our state contingent framework by introducing two nonmanipulability conditions.

The first extension which we call *No Advantageous Reallocation across Individuals* (*NARAI*) requires that no group of individuals benefits if transfers are allowed within a state. The next extension considers transfers across states which we call *No Advantageous Reallocation across States* (*NARAS*). We also use the axioms of *Anonymity* (*AN*), *Symmetry* (*SYM*), *Continuity* (*CONT*), *No Award for Null* (*NAN*), and *Independence* (*IND*). The *AN* axiom says that the rule should not differentiate between individuals according to their names. The *SYM* axiom requires that the names of the states do not matter. The *CONT* axiom states that the rule should be a continuous function in its arguments. The *NAN* axiom says that individuals with zero claims in all states should be allocated zero amount. The *IND* axiom says that if we mix two lotteries[2] with a third one, then the rationing rule associated with these two mixed lotteries does not depend on the third lottery used. We show that the ex-ante proportional rule is the only rule satisfying *NARAI, CONT*, *NAN*, and *NARAS* whereas the ex-post proportional rule is characterized by *NARAI, CONT*, *NAN*, *SYM*, and *IND*.

Another important aspect of this problem is to compare the shares allocated by the ex-ante proportional rule and the ex-post proportional rule. We compare shares given by the ex-ante and ex-post proportional rules for various distributions of claims and find sufficient conditions under which a particular individual will be favoured by one rule compared to the other. In particular we find that an individual with a deterministic claim always prefers the ex-post proportional rule over the ex-ante proportional rule.

In Section 2, we introduce the preliminaries. Section 3 presents the comparison between the ex-ante and the ex-post proportional rules. In Section 4, we provide our characterization results, and Section 5 concludes with some directions for future research.

2 Preliminaries

In the state-contingent claims framework, a rationing problem is defined as (N, S, x, p, t) where N is a finite set of individuals and S is a finite set of the states of the world.[3] The state contingent claim matrix $x \in \mathbb{R}_+^{N \times S}$ represents the claims of individuals in various states, where x_{is} denotes the claim of individual i in state s. The probabilities of states is denoted by common prior $p \in \Delta^{|S|-1}$ and $t \geq 0$ is the resource to be shared among the individuals.[4] It is assumed that $\sum\limits_{i \in N} x_{is} \geq t$ for all $s \in S$. Throughout the paper, we consider a fixed population N and a fixed set S of states. For the sake of brevity, we denote our problem

[2] By lottery we mean probability distribution over states of the world to be realized in the stage two.

[3] The standard rationing problem is defined as (N, x, t) where N is a finite set of agents, x is a claim vector $x = (x_i)_{i \in N} \geq 0$ such that $\sum\limits_{i \in N} x_i \geq t$ and $t \geq 0$ is the resource to be shared among the agents. A rationing rule φ assigns a vector of shares $\varphi(N, x, t) \in \mathbb{R}_+^N$ to every rationing problem such that $\sum\limits_{i \in N} \varphi_i(N, x, t) = t$.

[4] $\Delta^{|S|-1}$ denotes a $|S| - 1$ dimensional simplex.

(x,p,t). A non-empty set of problems is called a domain and is denoted by $\mathcal{D}$.[5] A rationing rule $\varphi:\mathcal{D}\to\mathbb{R}_+^N$ gives a vector of shares such that $\sum\limits_{i\in N}\varphi_i(x,p,t)=t$. Our main characterization results are obtained for rich domains which we define as follows:

Definition 1 *A domain $\bar{\mathcal{D}}$ is* rich *if for all $x,x'\in\mathbb{R}_+^{N\times S}$, for all $s\in S$, for all $p\in\Delta^{|S|-1}$, for all $t\geq 0$ with $x_{Ns}={x'_{Ns}}^6$, then $\{(x,p,t)\in\bar{\mathcal{D}}\Rightarrow(x',p,t)\in\bar{\mathcal{D}}\}$.*

Now we will define two rationing rules which involve the proportionality idea. Because our rules are based on proportionality idea, let us recall the standard proportional rule when we restrict our attention to just one state $s\in S$. Let us denote $x_{|s}=(x_{1s},x_{2s},...,x_{|N|s})\in\mathbb{R}_+^N$ as the vector of claims of individuals in state $s\in S$.

The proportional rule (pr^s) at state s is defined as a function $pr^s:\mathbb{R}_+^N\times\mathbb{R}_+\to\mathbb{R}_+$ where,

$$pr_i^s(x_{|s},t)=\frac{x_{is}}{x_{Ns}}t,\text{ for all } i\in N.$$

The ex-ante proportional rule ($\overline{pr}$) is defined by applying the proportional rule to the expectation of the state contingent claims.

$$\overline{pr_i}(x,p,t):=\frac{\sum\limits_{s\in S}(p_sx_{is})}{\sum\limits_{j\in N}\sum\limits_{s\in S}(p_sx_{js})}t=\frac{\sum\limits_{s\in S}(p_sx_{is})}{\sum\limits_{s\in S}(p_sx_{Ns})}t,\text{ for all } i\in N.$$

The ex-post proportional rule ($\widetilde{pr}$) is defined by expectation of the shares found by applying the proportional rule on the state contingent claims.

$$\widetilde{pr}_i(x,p,t):=\sum_{s\in S}\left(p_s\frac{x_{is}}{x_{Ns}}\right)t,\text{ for all } i\in N.$$

The illustration of the ex-ante proportional rule and the ex-post proportional rule for a simple economy with two people and two states is presented in Figure 1.

[5] More precisely this is a restricted domain of problems where N and S are fixed so a better notation would be $\mathcal{D}(N,S)$. However, for notational simplicity we use $\mathcal{D}$ since it does not raise any confusion.

[6] We use the notation $x_{Ts}:=\sum_{i\in T}(x_{is})$, where $T\subseteq N$.

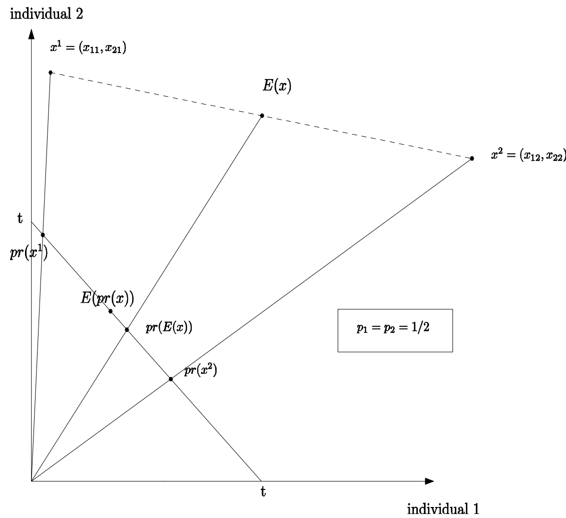

Figure 1: The ex-ante vs. the ex-post proportional rule

As shown in Figure 1, the ex-ante proportional rule and the ex-post proportional rule do not necessarily coincide. Therefore it would be interesting to know the conditions under which the shares according to two rules differ. For example, we would like to investigate which individuals prefer one rule over another. In the next section we will provide various comparisons between these two rules.

3 Comparisons

As we just witnessed, the shares allocated by the two rules can be different. These rules coincide only in special cases of state probabilities and claim vectors. In the following proposition we provide a general expression for the difference in shares by the two rules.

Proposition 1 *Let $(x, p, t) \in \mathcal{D}$ be given. Further assume that $x \in \mathbb{R}^{N \times S}_{++}$. The difference between the shares of an individual $i \in N$ given by the ex-ante proportional rule and the ex-post proportional rule is the following*

$$\overline{pr}_i(x,p,t) - \widetilde{pr}_i(x,p,t) = \frac{\sum\limits_{s \in S} \left[p_s x_{is} \left(1 - p_s - \sum\limits_{j \in N} \frac{x_{js}}{x_{is}} \sum\limits_{r \neq s} \frac{p_r x_{ir}}{x_{Nr}} \right) \right]}{\sum\limits_{s \in S} (p_s x_{Ns})} t.$$

Proof. *Define* $\alpha_{js} = \frac{x_{js}}{x_{is}}$ *for all* $j \in N$ *and for all* $s \in S$.
So the ex-ante proportional rule for individual i is given by
$\overline{pr}_i(x,p,t) = \frac{\sum\limits_{s\in S}(p_s x_{is})}{\sum\limits_{s\in S}(p_s x_{Ns})} t = \frac{\sum\limits_{s\in S}(p_s x_{is})}{\sum\limits_{s\in S}(p_s x_{is}\alpha_{Ns})} t.$
And the ex-post proportional rule for individual i is given by
$\widetilde{pr}_i(x,p,t) = \sum\limits_{s\in S}\left(p_s \frac{x_{is}}{x_{Ns}}\right) t = \sum\limits_{s\in S} \frac{p_s}{\alpha_{Ns}} t.$
So the difference between the ex-ante proportional rule and the ex-post proportional rule is

$$\overline{pr}_i(x,p,t) - \widetilde{pr}_i(x,p,t) = \frac{\sum\limits_{s\in S}(p_s x_{is})}{\sum\limits_{s\in S}(p_s x_{is}\alpha_{Ns})} t - \sum\limits_{s\in S}\frac{p_s}{\alpha_{Ns}} t =$$

$$= \frac{\sum\limits_{s\in S}(p_s x_{is}) \prod\limits_{s\in S}\alpha_{Ns} - \sum\limits_{s\in S}(p_s x_{is}\alpha_{Ns}) \prod\limits_{s\in S}\alpha_{Ns} \sum\limits_{s\in S}\frac{p_s}{\alpha_{Ns}}}{\prod\limits_{s\in S}\alpha_{Ns} \sum\limits_{s\in S}(p_s x_{is}\alpha_{Ns})} t =$$

$$= \frac{\prod\limits_{s\in S}\alpha_{Ns}\left[\sum\limits_{s\in S}(p_s x_{is}) - \sum\limits_{s\in S}[p_s x_{is}\alpha_{Ns}] \sum\limits_{s\in S}\frac{p_s}{\alpha_{Ns}}\right]}{\prod\limits_{s\in S}\alpha_{Ns}\sum\limits_{s\in S}(p_s x_{is}\alpha_{Ns})} t =$$

$$= \frac{\sum\limits_{s\in S}\left[\left(p_s - p_s^2\right)x_{is}\right] - \sum\limits_{s\in S}(p_s x_{is}\alpha_{Ns}) \sum\limits_{r\neq s}\frac{p_r}{\alpha_{Nr}}}{\sum\limits_{s\in S}(p_s x_{is}\alpha_{Ns})} t = \frac{\sum\limits_{s\in S}\left[p_s x_{is}\left(1 - p_s - \sum\limits_{j\in N}\alpha_{js}\sum\limits_{r\neq s}\frac{p_r}{\alpha_{Nr}}\right)\right]}{\sum\limits_{s\in S}(p_s x_{is}\alpha_{Ns})} t =$$

$$\frac{\sum\limits_{s\in S}\left[p_s x_{is}\left(1 - p_s - \sum\limits_{j\in N}\frac{x_{js}}{x_{is}}\sum\limits_{r\neq s}\frac{p_r x_{ir}}{x_{Nr}}\right)\right]}{\sum\limits_{s\in S}(p_s x_{Ns})} t. \quad \blacksquare$$

We use this expression for two-state and two-individual scenario in the following example to illustrate the conditions on claim vectors such that these two rules coincide.

Example 1 *For* $|N| = |S| = 2$ *and* $p_1 = p_2 = \frac{1}{2}$, *we have*

$$\overline{pr}_1(x,p,t) - \widetilde{pr}_1(x,p,t) = \frac{x_{11}x_{12}\left[\left(\frac{x_{22}}{x_{12}} - \frac{x_{21}}{x_{11}}\right)(x_{11} + x_{21} - (x_{12} + x_{22}))\right]}{2(x_{12} + x_{22})(x_{11} + x_{21})(x_{11} + x_{21} + x_{12} + x_{22})} t.$$

For $|N| = 2$, both rules give identical shares when the sum of the claims is equal for each state (Figure 2) or the ratio of the claims for each state is equal (Figure 3).

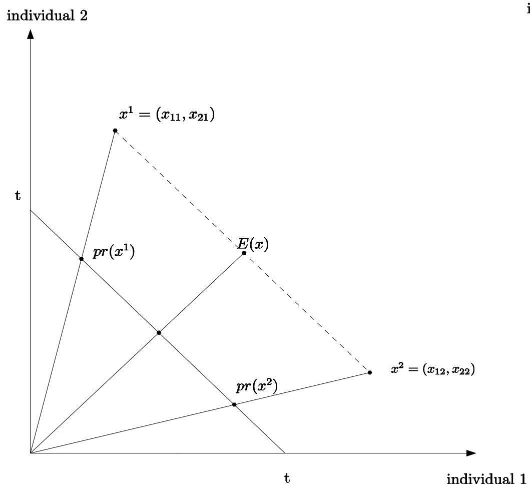

Figure 2: The sum of the claims for each state is equal: $x_{N1} = x_{N2}$

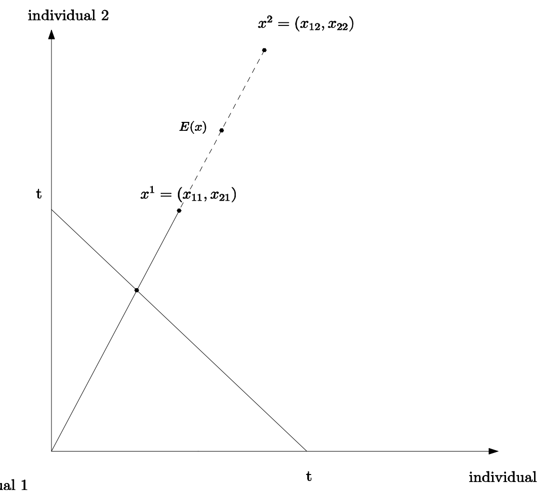

Figure 3: The ratio of the claims for each state is equal: $\frac{x_{11}}{x_{21}} = \frac{x_{12}}{x_{22}}$

The next proposition will illustrate that if an individual has a deterministic claim, then he would prefer the ex-post proportional rule over the ex-ante proportional rule. Conversely, the ex-ante proportional rule protects an individual whose claim distribution has a higher spread.

Proposition 2 *For all $(x, p, t) \in \mathcal{D}$ and for all $i \in N$, If $x_{is} = c \geq 0$ for all $s \in S$, then $\widetilde{pr}_i(x, p, t) \geq \overline{pr}_i(x, p, t)$.*

Proof. $\widetilde{pr}_i(x,p,t)-\overline{pr}_i(x,p,t) = \sum\limits_{s\in S}\left(p_s \frac{c}{x_{Ns}}\right)t - \frac{\sum_{s\in S}(p_s c)}{\sum_{s\in S}(p_s x_{Ns})}t = \left(\sum\limits_{s\in S}\frac{p_s}{x_{Ns}} - \frac{1}{\sum_{s\in S}(p_s x_{Ns})}\right)ct =$
$\frac{\sum_{s\in S} p_s^2 + \sum_{s\neq t}\left(p_s p_t\left(\frac{x_{Ns}}{x_{Nt}}+\frac{x_{Nt}}{x_{Ns}}\right)\right)-1}{\sum_{s\in S}(p_s x_{Ns})}ct = \frac{1-2\sum_{s\neq t}(p_s p_t)+\sum_{s\neq t}\left(p_s p_t\left(\frac{x_{Ns}}{x_{Nt}}+\frac{x_{Nt}}{x_{Ns}}\right)\right)-1}{\sum_{s\in S}(p_s x_{Ns})}ct =$
$= \frac{\sum_{s\neq t}\left(p_s p_t\left(\frac{x_{Ns}}{x_{Nt}}+\frac{x_{Nt}}{x_{Ns}}\right)-2\right)}{\sum_{s\in S}(p_s x_{Ns})}ct = \frac{\sum_{s\neq t}\left(p_s p_t\frac{(x_{Ns}-x_{Nt})^2}{x_{Ns}x_{Nt}}\right)}{\sum_{s\in S}(p_s x_{Ns})}ct \geq 0.$ ■

4 Characterizations

In this section we introduce some axioms and we use them to characterize the rules we have discussed above.

Continuity (CONT): For all $(x, p, t) \in \mathcal{D}$ and for all sequences $(x^k, p^k, t^k) \in \mathcal{D}$, if $(x^k, p^k, t^k) \to (x, p, t)$, then $\varphi(x^k, p^k, t^k) \to \varphi(x, p, t)$.

Continuity tells us that small changes in the parameters of the problem do not bring big jumps in the shares. Continuity is desirable because we do not want small errors (e.g., measurement errors) to lead to big changes in the shares.

Anonymity (AN): For all $(x, p, t) \in \mathcal{D}$, for all permutations $\sigma : N \to N$, and for all $i \in N$, $\varphi_i(x, p, t) = \varphi_{\sigma(i)}(x^\sigma, p, t)$, where $x^\sigma_{is} = x_{\sigma(i)s}$.

Anonymity says that the names of the individuals do not matter. This is a very natural axiom and is central to the literature on fairness.

Symmetry (SYM): For all $(x, p, t) \in \mathcal{D}$, for all permutations $\rho : S \to S$, and for all $i \in N$, $\varphi_i(x, p, t) = \varphi_i(x^\rho, p^\rho, t)$, where $x^\rho_{is} = x_{i\rho(s)}$ and $p^\rho_s = p_{\rho(s)}$.

Symmetry is similar to the Anonymity Axiom with the role of individuals being substituted by states. It says that the names of the states do not matter.

No Award for Null Players (NAN): For all $(x, p, t) \in \mathcal{D}$ and for all $i \in N$, if $x_{is} = 0$ for all $s \in S$, then $\varphi_i(x, p, t) = 0$.

No Award for Null Players Axiom says that an individual with zero claim for each state should get zero share. This axiom is also called the Dummy axiom or Null axiom in the literature.

Moulin (1985) defined the No Advantageous Reallocation axiom to characterize the egalitarian and the utilitarian solutions in quasi-linear social choice problems. We will define two axioms on invariance to reallocation in a similar manner where transfers are made either across individuals or across states.

No Advantageous Reallocation across Individuals (NARAI): For all $(x, p, t), (x', p, t) \in \mathcal{D}$ and for all $i \in N$, if $\sum_{j \in N \setminus \{i\}} x_{js} = \sum_{j \in N \setminus \{i\}} x'_{js}$ and $x_{is} = x'_{is}$ for all $s \in S$, then $\varphi_i(x, p, t) = \varphi_i(x', p, t)$.

NARAI states that the share of individual i depends on the sum of the claims of the individuals other than himself. In other words, individuals other than i cannot affect the share of i by reallocating their claims among themselves, i.e. the share of individual i is a function of $x_i, x_{N \setminus i}, p$, and t.

No Advantageous Reallocation across States (NARAS): For all $(x,p,t),(x',p,t) \in \mathcal{D}$ and for all $i \in N$, if $\sum_{s\in S} x_{is} = \sum_{s\in S} x'_{is}$ such that $\sum_{s\in S}(p_s x_{is}) = \sum_{s\in S}(p_s x'_{is})$ and $x_{js} = x'_{js}$ for all $j \in N\backslash\{i\}$ and for all $s \in S$, then $\varphi_j(x,p,t) = \varphi_j(x',p,t)$ for all $j \in N\backslash\{i\}$.

NARAS implies that if individual i reallocates his claim across all the states provided that his expected claim remains constant then other individuals' share (hence his own share) will not change. This axiom becomes compelling when we want the shares of individuals to be invariant to the distribution of the claims. Moreover, NARAS has a flavor of the standard non-bossy axiom which requires the shares of all the other agents to remain unchanged if an agent unilaterally changes his report in a way that does not affect his own share.[7]

For the remainder of the paper we will consider rich domains, $\bar{\mathcal{D}}$. Before we characterize our two rules, namely the ex-post proportional rule and the ex-ante proportional rule, we characterize a general class of rules that satisfy NARAI, Anonymity, and Continuity. This class of rules includes both the ex-ante proportional rule and the ex-post proportional rule. Moreover, this class includes rules like equal split, and non-symmetric proportional rules among others.

Theorem 1 *Let $|N| \geq 3$ and $(x,p,t) \in \bar{\mathcal{D}}$. A rationing rule φ satisfies NARAI, CONT, and AN if and only if there exists a continuous function $W_s : \mathbb{R}^S \times \Delta^{|S|-1} \times \mathbb{R}_+ \to \mathbb{R}$ for all $s \in S$ such that for all $i \in N$ we have*

$$\varphi_i(x,p,t) = \frac{t}{|N|} + \sum_{s\in S}\left[\left(x_{is} - \frac{x_{Ns}}{|N|}\right) W_s(x_N,p,t)\right]. \tag{1}$$

Proof. The "if" part of the statement is obvious. We will prove the "only if" part. Let $(x,p,t) \in \bar{\mathcal{D}}$. Let φ be a rationing rule satisfying NARAI, CONT, and AN.

Let $x' = (x_1 + x_2, 0, x_3, ...)$. Applying NARAI for every individual belonging to $N\backslash\{1,2\}$, we get $\sum_{i\in N\backslash\{1,2\}} \varphi_i(x,p,t) = \sum_{i\in N\backslash\{1,2\}} \varphi_i(x',p,t)$. Therefore

$$\varphi_1(x,p,t) + \varphi_2(x,p,t) = \varphi_1(x',p,t) + \varphi_2(x',p,t). \tag{2}$$

[7] Note that the standard non-bossy axioms protect the other individuals from any unilateral change of report by an individual whereas NARAS axiom protects those individuals only against a specific change of the report, i.e., the reallocated report with the same expected value.

Let $x^{''} = (x_1, x_{N\setminus\{1\}}, 0, 0, ...)$. Now we apply NARAI for individual 1. So $\varphi_1(x,p,t) = \varphi_1(x'',p,t)$. This implies

$$\varphi_{N\setminus\{1\}}(x,p,t) = \varphi_{N\setminus\{1\}}(x^{''},p,t). \tag{3}$$

$$\left.\begin{array}{l} \text{By (3) and AN, } \varphi_2(x,p,t) = \varphi_1(x_2, x_{N\setminus\{2\}}, 0, ..., 0, p, t). \\ \text{By (3), } \varphi_1(x',p,t) = \varphi_1(x_1 + x_2, x_{N\setminus\{1,2\}}, 0, ..., 0, p, t). \\ \text{By (3) and AN, } \varphi_2(x',p,t) = \varphi_1(0, x_N, 0, ..., 0, p, t). \end{array}\right\} \tag{4}$$

Let us plug (4) into (2) to get
$\varphi_1(x_1, x_{N\setminus\{1\}}, 0, ..., 0, p, t) + \varphi_1(x_2, x_{N\setminus\{2\}}, 0, ..., 0, p, t) = \varphi_1(x_1 + x_2, x_{N\setminus\{1,2\}}, 0, ..., 0, p, t) + \varphi_1(0, x_N, 0, ..., 0, p, t)$
Let us define $f : \mathbb{R}^S_+ \times \mathbb{R}^S \times \Delta^{|S|-1} \times \mathbb{R}_+ \to \mathbb{R}$ such that
$f(x_i, x_N, p, t) = \varphi_1(x_i, x_{N\setminus\{i\}}, 0, ..., 0, p, t) - \varphi_1(0, x_N, 0, ..., 0, p, t)$
and define $g : \mathbb{R}^S \times \Delta^{|S|-1} \times \mathbb{R}_+ \to \mathbb{R}$ such that $g(x_N, p, t) = \varphi_1(0, x_N, 0, ..., 0, p, t)$.
Thus we get

$$f(x_1, x_N, p, t) + f(x_2, x_N, p, t) = f(x_1 + x_2, x_N, p, t). \tag{5}$$

As it is evident from (5), f is additive in the first term and by definition f is continuous (since φ is continuous). So by invoking Cor 3.1.9, p.51, from Eichhorn (1978), we deduce that f is linear in the first term, that is, there exists a continuous function $W : \mathbb{R}^S \times \Delta^{|S|-1} \times \mathbb{R}_+ \to \mathbb{R}^S$ such that
$f(x_i, x_N, p, t) = \sum_{s\in S} [W_s(x_N, p, t) x_{is}]$.
So $\varphi_i(x,p,t) = \sum_{s\in S} [(W_s(x_N, p, t) x_{is})] + g(x_N, p, t)$.
Summing over $i \in N$ we get
$\sum_{i\in N} \varphi_i(x,p,t) = \sum_{s\in S} [(W_s(x_N, p, t) x_{Ns})] + |N| g(x_N, p, t) = t$.
So $g(x_N, p, t) = \frac{t - \sum_{s\in S} [(W_s(x_N,p,t) x_{Ns})]}{|N|}$.
Hence we get the desired functional form.
$\varphi_i(x,p,t) = \frac{t}{|N|} + \sum_{s\in S} \left[\left(x_{is} - \frac{x_{Ns}}{|N|}\right) W_s(x_N, p, t)\right]$, for all $i \in N$. ■

Remark 1 *Our axioms NARAI, CONT, and AN are independent. To show independence of the axioms we provide the following examples:*

- $\varphi_i(x,p,t) = \min\left\{\lambda, \sum_{s\in S} (p_s x_{is})\right\}$ where $\sum_{i\in N} \min\left\{\lambda, \sum_{s\in S} (p_s x_{is})\right\} = t$.
  This ex-ante uniform gains rule satisfies all the axioms except NARAI.

- $\varphi_i(x,p,t) = \begin{cases} \overline{pr}_i(x,p,t) & \text{if } \sum\limits_{s\in S} x_{Ns} < 2t|S| \\ \widetilde{pr}_i(x,p,t) & \text{o/w} \end{cases}$.

  This rule satisfies all the axioms except CONT.

- $\varphi(x,p,t) = (t,0,0,...,0)$ for all $(x,p,t) \in \bar{\mathcal{D}}$. This priority rule satisfies all the axioms except AN.

The family of rules characterized in the theorem above contains various rules including proportional and egalitarian rules. In the example below, we provide some notable rules that belong to this family.

Example 2 *Various weight functions $W_s(x_N,p,t)$ give rise to various rules. Some of the examples are:*

- Equal split rule, i.e., $\varphi_i(x,p,t) = \frac{t}{|N|}$, when $W_s(x_N,p,t) = 0$.
- When $W_s(x_N,p,t)$ satisfies $\sum\limits_{s\in S}[W_s(x_N,p,t)x_{Ns}] = t$, we get the family of proportional rules, i.e., $\varphi_i(x,p,t) = \sum\limits_{s\in S}[W_s(x_N,p,t)x_{is}]$.
- When the weight functions are uniform with respect to states, that is, $W_s(x_N,p,t) = \frac{t}{\sum\limits_{s\in S} x_{Ns}}$ for all s, we get $\varphi_i(x,p,t) = \frac{\sum\limits_{s\in S} x_{is}}{\sum\limits_{s\in S} x_{Ns}} t$.
- The ex-ante proportional rule, $\varphi_i(x,p,t) = \frac{\sum\limits_{s\in S}(p_s x_{is})}{\sum\limits_{s\in S}(p_s x_{Ns})} t$ when $W_s(x_N,p,t) = \frac{p_s t}{\sum\limits_{s\in S}(p_s x_{Ns})}$.
- The ex-post proportional rule, $\varphi_i(x,p,t) = \sum\limits_{s\in S}\left(p_s \frac{x_{is}}{x_{Ns}}\right) t$ when $W_s(x_N,p,t) = \frac{p_s t}{x_{Ns}}$.
- The family contains non-symmetric proportional rules with respect to states as well, e.g., $\varphi_i(x,p,t) = \frac{x_{i1}}{x_{N1}} t$ when $W_1(x_N,p,t) = \frac{t}{x_{N1}}$, $W_2(x_N,p,t) = W_3(x_N,p,t) = ... = 0$ (all the weight is given to state 1).

In Theorem 1 we characterized the family of rules which include both the ex-ante proportional rule and the ex-post proportional rule. We provide characterization of these rules in Theorems 3 and 4. Before characterizing our two rules, we present a family of generalized proportional rules in Theorem 2 that satisfies NARAI, CONT, and NAN.

Theorem 2 *Let $|N| \geq 3$ and $(x,p,t) \in \bar{\mathcal{D}}$. A rationing rule φ satisfies NARAI, CONT and NAN if and only if there exists a continuous $W_s : \mathbb{R}^S \times \Delta^{|S|-1} \times \mathbb{R}_+ \to \mathbb{R}$ for all $s \in S$ such that for all $i \in N$, we have*

$$\varphi_i(x,p,t) = \sum_{s \in S} \left[(W_s(x_N,p,t)x_{is}) \right].$$

Proof. The "if" part is obvious. We will prove the "only if" part. Let $(x,p,t) \in \bar{\mathcal{D}}$. Let φ be a rationing rule satisfying NARAI, CONT, and NAN. First we establish that NARAI and NAN imply AN. Since φ satisfies NARAI, we know from Theorem 1 of Ju et al. (2007) that $\varphi_i(x,p,t)$ must be of the form

$$\varphi_i(x,p,t) = A_i(x_N,p,t) + \sum_{s \in S} \hat{W}_s(x_{is}, x_N, p, t),$$

where $A : \mathbb{R}^S_+ \times \Delta^{|S|-1} \times \mathbb{R}_+ \to \mathbb{R}^N$ and $\hat{W} : \mathbb{R}_+ \times \mathbb{R}^S_+ \times \Delta^{|S|-1} \times \mathbb{R}_+ \to \mathbb{R}^S$, and for all $s \in S$, $\hat{W}_s(., x_N, p, t)$ is additive. Applying NAN, we get $A_i(x_N,p,t) = 0$ for all $(x,p,t) \in \bar{\mathcal{D}}$. Therefore,

$$\varphi_i(x,p,t) = \sum_{s \in S} \hat{W}_s(x_{is}, x_N, p, t),$$

which ensures that AN is satisfied.
Thus we know that φ satisfies the premises of our Theorem 1. Hence, we have

$$\varphi_i(x,p,t) = \frac{t}{|N|} + \sum_{s \in S} \left[\left(x_{is} - \frac{x_{Ns}}{|N|} \right) W_s(x_N,p,t) \right].$$

Now we apply NAN to get the desired functional form of $\varphi_i(x,p,t)$. Take $i \in N$ with $x_{is} = 0$ for all $s \in S$. NAN implies that $\varphi_i(x,p,t) = \frac{t}{|N|} + \sum_{s \in S} \left[\left(x_{is} - \frac{x_{Ns}}{|N|} \right) W_s(x_N,p,t) \right] = 0$. Therefore, we must have $\sum_{s \in S} \left[W_s(x_N,p,t)x_{Ns} \right] = t$. Hence we get the desired result, $\varphi_i(x,p,t) = \sum_{s \in S} \left[(W_s(x_N,p,t)x_{is}) \right]$. Note that the general functional form of φ_i must hold for any problem $(x,p,t) \in \bar{\mathcal{D}}$ including those $(x,p,t) \in \bar{\mathcal{D}}$ where there exists $i \in N$ such that $x_{is} = 0$ for all $s \in S$.
■

Remark 2 *For $|N| = 2$, NARAI is trivially satisfied. In order to get our characterization, we can use null consistency axiom in a variable population setting similar to Chun (1988). This axiom states that if an individual i claims zero for* *each state, then shares of the individuals other than i are invariant of whether individual i is present or not. Notice that this axiom implies NAN. By replacing NAN with Null Consistency one would obtain the desired characterization.*

Remark 3 *Our axioms NARAI, CONT, and NAN are independent. To show independence of the axioms we provide the following examples:*

- $\varphi_i(x,p,t) = \min\left\{\lambda, \sum\limits_{s\in S}(p_s x_{is})\right\}$ where $\sum\limits_{i\in N}\min\left\{\lambda, \sum\limits_{s\in S}(p_s x_{is})\right\} = t.$ This ex-ante uniform gains rule satisfies all the axioms except NARAI.

- $\varphi_i(x,p,t) = \begin{cases} \overline{pr}_i(x,p,t) & \text{if } \sum\limits_{s\in S} x_{Ns} < 2t|S| \\ \widetilde{pr}_i(x,p,t) & \text{o/w} \end{cases}.$

  This rule satisfies all the axioms except CONT.

- $\varphi_i(x,p,t) = \frac{t}{|N|}$ for all $i \in N$ and for all $(x,p,t) \in \bar{\mathcal{D}}$. This equal split rule satisfies all the axioms except NAN.

The ex-ante proportional rule allocates the resource to the individuals in proportion to their expected claims. Due to the simplicity of this rule, it is practical and thus appealing in various scenarios. The key advantage of this rule is that rather than requiring the planner to have information about the whole distribution of the claims, it suffices for the planner to elicit expected claims. In many cases, it is impossible for the individuals to know the exact distribution of their claims beforehand. For example, individuals may not know their precise claims for some improbable events. They are more likely to know, perhaps from historical experience, an estimate of their expected claim. In Theorem 3, we characterize the ex-ante proportional rule by using NARAS in addition to the axioms of Theorem 2.

Theorem 3 *Let $|N| \geq 3$, $|S| \geq 3$, and $(x,p,t) \in \bar{\mathcal{D}}$. A rationing rule φ satisfies NARAI, CONT, NAN, and NARAS if and only if φ is the ex-ante proportional rule.*

Proof. The "if" part is obvious. We will prove the "only if" part. Let $(x,p,t) \in \bar{\mathcal{D}}$. Let φ be a rationing rule satisfying NARAI, CONT, NAN, and NARAS. Given that φ satisfies the premises of Theorem 2, we have $\varphi_i(x,p,t) = \sum\limits_{s\in S}[(W_s(x_N,p,t)x_{is})]$, for all $i \in N$.

Let $x' \in \mathbb{R}_+^{N\times S}$ such that $\sum_{s\in S} x_{is} = \sum_{s\in S} x'_{is}$ and $\sum_{s\in S}(p_s x_{is}) = \sum_{s\in S}(p_s x'_{is})$ and $x_{js} = x'_{js}$, for all $j \in N\backslash\{i\}$ and for all $s \in S$. So we get

$$\sum_{s\in S}\left[p_s\left(x_{is} - x'_{is}\right)\right] = 0, \text{ for all } i \in N. \tag{6}$$

By NARAS, we have $\varphi_i(x,p,t) = \varphi_i(x',p,t)$, for all $i \in N$. Then

$$\sum_{s\in S}\left[(W_s(x_N,p,t)x_{is})\right] = \sum_{s\in S}\left[(W_s(x'_N,p,t)x'_{is})\right], \text{ for all } i \in N. \tag{7}$$

Fix $j \in N\backslash\{i\}$, by (7) we have $\sum_{s\in S}[(W_s(x_N,p,t)x_{js})] = \sum_{s\in S}\left[\left(W_s(x'_N,p,t)x'_{js}\right)\right] = \sum_{s\in S}[(W_s(x'_N,p,t)x_{js})]$.

By the richness of $\bar{\mathcal{D}}$, we have $W_s(x_N,p,t) = W_s(x'_N,p,t)$, for all $s \in S$. By using (7) we get

$$\sum_{s\in S}\left[W_s(x_N,p,t)\left(x_{is} - x'_{is}\right)\right] = 0, \text{ for all } i \in N. \tag{8}$$

By (6) and (8), we deduce that p and W are colinear. So for all $s \in S$ there exists $h_s : \mathbb{R}^S \times \Delta^{|S|-1} \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $W_s(x_N,p,t) = h_s(x_N,p,t)p_s$ for all $s \in S$. Moreover by NARAS, we must have $h_s = h_{s'}$ for all $s' \neq s$. To see this, consider two states, say s and s' such that $h_s(x_N,p,t) > h_{s'}(x_N,p,t)$ for some $(x,p,t) \in \bar{\mathcal{D}}$. In this case, an individual i would have an advantageous mean preserving reallocation. So we can write $h_s(x_N,p,t)$ as $h(x_N,p,t)$.

Summing over $i \in N$, $\sum_{i\in N}\sum_{s\in S}[(W_s(x_N,p,t)x_{is})] = \sum_{i\in N}\sum_{s\in S}[h(x_N,p,t)p_s x_{is}] = h(x_N,p,t)\sum_{s\in S}(p_s x_{Ns}) = t$.

So $h(x_N,p,t) = \frac{t}{\sum_{s\in S}(p_s x_{Ns})}$. And $\varphi_i(x,p,t) = \sum_{s\in S}[h(x_N,p,t)p_s x_{is}] = \frac{\sum_{s\in S}(p_s x_{is})}{\sum_{s\in S}(p_s x_{Ns})}t$, for all $i \in N$. ■

Remark 4 *Note that for $|S| = 2$, mean preserving reallocation is only possible when $p_1 = p_2$ which makes NARAS ineffective to obtain $h_s = h_{s'}$ for all $s' \neq s$. One can add SYM to obtain the desired characterization.*

Before characterizing the ex-post proportional rule, we note that NARAS is not satisfied by the ex-post proportional rule. Figure 4 below illustrates an instance where NARAS is violated using a two-individual and two-state example where both states are equally likely. Here, individual 2 has a deterministic claim, i.e., he claims c_2 in both states. Now, if individual 1 also has a deterministic claim of, say c_1, in both states then the final allocation by the ex-post proportional rule is given by $pr(E(x))$. In contrast, suppose individual 1 reallocates his claims across the two states (as x_{11} and x_{12}) in such a way that the mean of the claims is preserved at c_1. In this case, the final allocation by the ex-post proportional rule is given by $E(pr(x))$. Clearly, individual 1 is worse off by this mean-preserving spread and thus NARAS is violated.

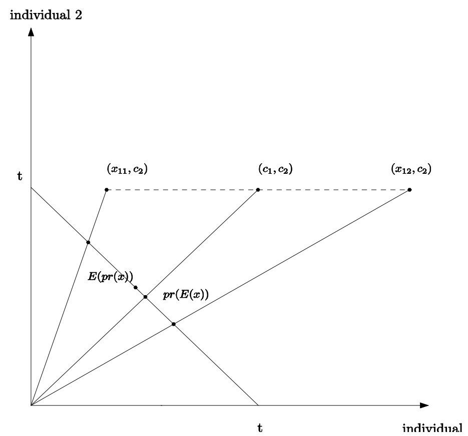

Figure 4: Ex-post proportional rule violates NARAS: Under ex-post proportional rule, mean-preserving spread makes agent 1 worse off when agent 2 has a deterministic claim.

The following examples illustrate that the rest of the axioms in our theorem are independent as well.

Remark 5 *Our axioms NARAI, CONT, NAN, and NARAS are independent. To show independence of the axioms we provide the following examples:*

- $\varphi_i(x,p,t) = \min\left\{\lambda, \sum_{s\in S}(p_s x_{is})\right\}$ where $\sum_{i\in N}\min\left\{\lambda, \sum_{s\in S}(p_s x_{is})\right\} = t$. This ex-ante uniform gains rule satisfies all the axioms except NARAI.

- $\varphi_i(x,p,t) = \begin{cases} \overline{pr}_i(x,p,t) & \text{if } \sum_{s\in S} x_{Ns} < 2t|S| \\ \sum_{s\in S} \frac{x_{is}}{x_{Ns}} t & \text{o/w} \end{cases}.$

This rule satisfies all the axioms except CONT.

- $\varphi_i(x,p,t) = \frac{t}{|N|}$ for all $i \in N$ and for all $(x,p,t) \in \mathcal{D}$. This equal split rule satisfies all the axioms except NAN.
- $\varphi_i(x,p,t) = \widetilde{pr}_i(x,p,t)$. The ex-post proportional rule satisfies all the axioms except NARAS.

Now we characterize the ex-post proportional rule. The functional form of the ex-post proportional rule is additively separable with respect to the states. This is similar to the *expected utility form* due to von Neumann and Morgenstern (1944). Notice that the lotteries in our framework are analogous to probabilities of the states, $(p_s)_{s\in S}$ and the outcomes are given by $(x_{|s}, t)_{s\in S}$. Moreover the preference of agent i over lottery $p \in \Delta^{|S|-1}$ is defined by the ordering given by the rule $\varphi_i(x,p,t)$. Therefore, in the spirit of Expected Utility Theory, we will utilize the independence axiom which is defined below.

Independence (IND): For all $(x,p,t),(x,q,t),(x,r,t) \in \mathcal{D}$, for all $i \in N$, and for all $\lambda \in (0,1)$, we have $\varphi_i(x,p,t) \geq \varphi_i(x,q,t)$ if and only if $\varphi_i(x,\lambda p+(1-\lambda)r,t) \geq \varphi_i(x,\lambda q+(1-\lambda)r,t)$.

IND implies that the ordering of an individual's share with respect to two different state probabilities is preserved if these two state probabilities are mixed with any other state probability.

The ex-post proportional rule is obtained by first finding the shares of an individual for each state of the world using the proportional rule on the claim profile in that state and then taking expectation of the shares over all the states. Compared to the ex-ante proportional rule, which is simple and practical in some settings as we have discussed above, the ex-post proportional rule embodies a deeper sense of proportionality. This is so because the shares are found by applying proportional rule for each possible state of the world. Therefore, there is a strong case for using this rule based on this principle of proportionality if we are able to compute share of an individual for each possible state of the world. By adding SYM and IND to the axioms of Theorem 2, we get characterization of the ex-post proportional rule in Theorem 4.

Theorem 4 *Let $|N| \geq 3$, and $(x,p,t) \in \bar{\mathcal{D}}$. A rationing rule φ satisfies NARAI, CONT, NAN, SYM, and IND if and only if φ is the ex-post proportional rule.*

Proof. The "if" part is obvious. We will prove the "only if" part. Let $(x, p, t) \in \bar{\mathcal{D}}$. Let φ be a rationing rule satisfying NARAI, CONT, NAN, SYM, and IND. Given that φ satisfies the premises of Theorem 2, we have

$$\varphi_i(x, p, t) = \sum_{s \in S} \left[(W_s(x_N, p, t) x_{is}) \right], \text{ for all } i \in N. \tag{9}$$

Notice that the number of states is finite. Moreover, given the way we have defined preferences over lotteries, i.e., as the ordering given by $\varphi_i(x, p, t)$, our CONT and IND axioms imply that the preferences are continuous in p and satisfy v.N-M Independence axiom. Therefore we can utilize the Expected Utility Theorem and deduce that φ_i is additively separable with respect to probabilities. That is, for all $x \in \mathbb{R}_+^{N \times S}$ for all $p \in \Delta^{|S|-1}$, and for all $s \in S$ there exists $u_{is} : \mathbb{R}_+^N \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi_i(x, p, t) = \sum_{s \in S} \left[p_s u_{is} \left(x_{|s}, t \right) \right], \text{ for all } i \in N. \tag{10}$$

.

By SYM, we have $u_{is} = u_{is'}$ for all $s, s' \in S$. Therefore we have

$$\varphi_i(x, p, t) = \sum_{s \in S} \left[p_s u_i \left(x_{|s}, t \right) \right], \text{ for all } i \in N. \tag{11}$$

.

By (9) and (11) we deduce that for all $s \in S$ there exists $v : \mathbb{R}_+^2 \to \mathbb{R}_+$ such that

$$\varphi_i(x, p, t) = \sum_{s \in S} \left[p_s x_{is} v(x_{Ns}, t) \right], \text{ for all } i \in N. \tag{12}$$

Consider a degenerate lottery δ_s, that is, fix $s \in S$ and let $p_s = 1$.
So $\varphi_i(x, \delta_s, t) = x_{is} v_s(x_N, t)$.
Summing over $i \in N$, we get $\sum_{i \in N} \varphi_i(x, \delta_s, t) = \sum_{i \in N} [x_{is} v(x_{Ns}, t)] = v(x_{Ns}, t) x_{Ns} = t$. So $v(x_{Ns}, t) = \frac{t}{x_{Ns}}$. Hence we get the ex-post proportional rule.
$\varphi_i(x, p, t) = \sum_{s \in S} [p_s x_{is} v(x_{Ns}, t)] = \sum_{s \in S} \left(p_s \frac{x_{is}}{x_{Ns}} \right) t$. ■

Remark 6 *Our axioms NARAI, CONT, NAN, SYM, and IND are independent. To show independence of the axioms we provide the following examples:*

- $\varphi_i(x, p, t) = \sum_{s \in S} (p_s \min\{\lambda_s, x_{is}\})$ where λ_s is found by $\sum_{i \in N} \min\{\lambda_s, x_{is}\} = t$, for all s. This ex-post uniform gains rule satisfies all the axioms except NARAI.

- $\varphi_i(x,p,t) = \begin{cases} \widetilde{pr}_i(x,p,t) & \text{if } \sum\limits_{s\in S} x_{Ns} < 2t|S| \\ \sum\limits_{s\in S} \frac{x_{is}}{x_{Ns}} t & \text{o/w} \end{cases}$.

This rule satisfies all the axioms except CONT.

- $\varphi_i(x,p,t) = \frac{t}{|N|}$ for all $i \in N$ and for all $(x,p,t) \in \bar{\mathcal{D}}$. This equal split rule satisfies all the axioms except NAN.

- $\varphi_i(x,p,t) = \frac{x_{i1}}{x_{N1}} t$. This non-symmetric rule satisfies all the axioms except SYM.

- $\varphi_i(x,p,t) = \overline{pr}_i(x,p,t)$. The ex-ante proportional rule satisfies all the axioms except IND.

5 Conclusion

We study rationing problems where claims are state contingent. We introduce two extensions of the proportional rules in our framework – the ex-ante and the ex-post proportional rules. Applying the proportional rule to the expected claim gives the ex-ante proportional rule. The ex-post proportional rule is defined as the expectation of the shares given by the proportional rule for various states.

To characterize these rules we propose two extensions of No Advantageous Reallocation introduced by Moulin (1985). The first extension, *NARAI*, requires that no group of individuals benefits if transfers are allowed across individuals for each state. The second extension, *NARAS*, implies that an individual cannot change other individuals' shares (hence his own share) by reallocating his claim across the states while his expected claim is constant.

We characterize the ex-ante proportional rule by *NARAI* and *NARAS* combined with Continuity, and No Award for Null Players. To characterize the ex-post proportional rule, we borrow Independence axiom from the Expected Utility Theory. This axiom says that by mixing two lotteries with a third one, the rationing rule remains unaffected by the choice of the third lottery. By replacing *NARAS* with the Independence Axiom, and adding Symmetry, we obtain the characterization of the ex-post proportional rule.

We also compare the shares of the two rules by considering the difference in the shares of an individual allocated by the two rules as a function of the claim vector and state probabilities. We have demonstrated the conditions under which the two rules coincide. It is also shown that an individual with deterministic claim will prefer the ex-post proportional rule over the ex-ante proportional rule.

This paper leads us to some important issues to be considered for future research. One such issue is to find axiomatic characterizations of the extensions of other important rules, such as uniform gains and uniform losses. It will also be interesting to extend our framework to situations, where the resource itself is state contingent, and where the individuals have subjective probabilities.

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